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A Class of Topologically Free Locally Convex Spaces
and Related Group Hopf Algebras

By

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A Class of Topologically Free Locally Convex Spaces and Related Group Hopf Algebras

Kalman George Brauner, Jr.

September 1972

Abstract

A locally convex space E will be called p -reflexive provided the evaluation map from E to $(E)'$ is a topological linear isomorphism, where the dual space in each case is given the topology of precompact convergence. Let \mathcal{D} denote the category of p -reflexive spaces and continuous linear maps. \mathcal{D} is shown to be complete, cocomplete, and self-dual.

Let $Sets$ denote the category of sets and functions and $\nu: \mathcal{D} \rightarrow Sets$, the forgetful functor.

Theorem. *There exist functors $Hom: \mathcal{D}^{op} \times \mathcal{D} \rightarrow \mathcal{D}$ and $\otimes: \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$ such that (1) $\nu \circ Hom = hom$; and (2) for all $A, B, C \in ob\mathcal{D}$, $Hom(A \otimes B, C)$ is naturally isomorphic to $Hom(A, Hom(B, C))$.*

Thus \mathcal{D} can be extended to be a closed, symmetric, monoidal category in the sense of Eilenberg and Kelly.

Let \mathcal{T} denote the category of k -spaces and continuous maps, and let $c: \mathcal{T}^{op} \times \mathcal{D} \rightarrow Sets$ denote the functor such that if $X \in ob\mathcal{T}$ and $E \in ob\mathcal{D}$, $c(X, E)$ equals the set of continuous functions from X to E .

Theorem. *There exist functors $C: \mathcal{T}^{op} \times \mathcal{D} \rightarrow \mathcal{D}$ and $M: \mathcal{T} \rightarrow \mathcal{D}$ such that (1) $\nu \circ C = c$, and (2) for all $X \in ob\mathcal{T}$ and $E \in ob\mathcal{D}$, $C(X, E)$ is naturally isomorphic to $Hom(M(X), E)$.*

It is known that \mathcal{T} can be extended to be a closed, symmetric, monoidal category. Let $\star: \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$ denote the product functor.

Theorem. *If $X, Y \in ob\mathcal{T}$, then $M(X \star Y)$ is naturally isomorphic to $M(X) \otimes M(Y)$.*

M can be extended to be a "strong," symmetric, monoidal functor. M will transform any algebraic structure that occurs at the level of \mathcal{T} to an analogous algebraic structure at the level of \mathcal{D} .

Theorem. *Let \mathcal{A} and \mathcal{B} be categories such that either (a) $\mathcal{A} = k$ -spaces and $\mathcal{B} = \mathcal{D}$ -coalgebras; (b) $\mathcal{A} = k$ -monoids and $\mathcal{B} = \mathcal{D}$ -bialgebras; or (c) $\mathcal{A} = k$ -groups and $\mathcal{B} = \mathcal{D}$ -Hopf algebras. Then M can be regarded as a functor from \mathcal{A} to \mathcal{B} . Furthermore, there exists a functor $\Omega: \mathcal{B} \rightarrow \mathcal{A}$ such that if $G \in ob\mathcal{A}$ and $H \in ob\mathcal{B}$, then $Mor_{\mathcal{A}}(G, \Omega(H))$ is naturally isomorphic to $Mor_{\mathcal{B}}(M(G), H)$. M is an adjoint of Ω .*

Loosely speaking, a \mathcal{D} -algebra is a p -reflexive space with an associative multiplication and a unit; a \mathcal{D} -coalgebra is a p -reflexive space with a coassociative comultiplication and a counit; a \mathcal{D} -bialgebra is a p -reflexive space which simultaneously has the structure of a \mathcal{D} -algebra and a \mathcal{D} -coalgebra with the property that the comultiplication and counit maps are \mathcal{D} -algebra morphisms; and a \mathcal{D} -Hopf algebra is a \mathcal{D} -bialgebra which admits an antipode. k -monoids and k -groups are k -spaces which satisfy the axioms of topological monoids and topological groups respectively, except that the k -space product is used in place of the usual product.

To my wonderful wife, Joen,
without whose help, in so many different ways,
this dissertation would not be possible.

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Summary of major results.

A locally convex space E will be called p-reflexive (resp., p-complete) provided the evaluation map from E to $(E')'$ is a topological linear isomorphism (resp., bijection), where the dual space in each case has the topology of precompact convergence. Note that quasi-complete spaces are p-complete, and that barrelled, quasi-complete spaces are p-reflexive. Let \mathcal{D} denote the category of p-reflexive spaces and continuous linear maps. It is shown that \mathcal{D} is complete (i.e. limits exist in \mathcal{D}), cocomplete (i.e. colimits exist in \mathcal{D}), and self-dual (i.e. \mathcal{D} is equivalent to its dual category) (cf. 3.5 and 4.9).

Let Sets denote the category of sets and functions; let \mathcal{D}^{op} denote the opposite or dual category of \mathcal{D} ; and let $\text{hom}: \mathcal{D}^{\text{op}} \times \mathcal{D} \rightarrow \text{Sets}$ denote the functor such that if $A, B \in \text{ob } \mathcal{D}$, then $\text{hom}(A, B)$ equals the set of all \mathcal{D} -morphisms from A to B . Let $v: \mathcal{D} \rightarrow \text{Sets}$ denote the forgetful functor. In this situation, we have

Theorem (cf. 3.1 and 17.11). There exist functors $\text{Hom}: \mathcal{D}^{\text{op}} \times \mathcal{D} \rightarrow \mathcal{D}$ and $\otimes: \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$ such that (1) $v \circ \text{Hom} = \text{hom}$; and (2) for all $A, B, C \in \text{ob } \mathcal{D}$, $\text{Hom}(A \otimes B, C)$ is naturally isomorphic to $\text{Hom}(A, \text{Hom}(B, C))$.

This theorem contains the basic ingredients needed to prove that \mathcal{D} can be extended to be a closed, symmetric, monoidal category in the sense of Eilenberg and Kelly (cf. 23.8).

Recall that a k -space is a Hausdorff topological space with the property that a subset is closed if its intersection with each compact subset is closed. Let \mathcal{J} denote the category of k -spaces and continuous maps; let \mathcal{J}^{op} denote its opposite or dual category; and let $c: \mathcal{J}^{\text{op}} \times \mathcal{D} \rightarrow \text{Sets}$ denote the functor such that if $X \in \text{ob } \mathcal{J}$ and $E \in \text{ob } \mathcal{D}$, then $c(X, E)$ equals the set of all continuous functions from X to E .

Theorem (cf. 10.1 and 10.6). There exist functors $c: \mathcal{J}^{\text{op}} \times \mathcal{D} \rightarrow \mathcal{D}$ and $M: \mathcal{J} \rightarrow \mathcal{D}$ such that (1) $v \circ C = c$, and (2) for all $X \in \text{ob } \mathcal{J}$ and $E \in \text{ob } \mathcal{D}$, $C(X, E)$ is naturally isomorphic to $\text{Hom}(M(X), E)$.

Corollary (cf. 10.6). Let X be a k -space. Then there exists a continuous map $\varepsilon: X \rightarrow M(X)$ with the property that if E is a p -complete space and $f: X \rightarrow E$ is a continuous map, then there exists a unique continuous linear map $\bar{f}: M(X) \rightarrow E$ such that the following diagram commutes:

$$\begin{array}{ccc}
 X & \xrightarrow{f} & E \\
 \varepsilon \downarrow & \nearrow \bar{f} & \\
 M(X) & &
 \end{array}$$

It is known that products exist in \mathcal{J} . As a matter of fact, it is known that \mathcal{J} can be extended to be a closed, symmetric, monoidal category. Let $\mu: \mathcal{J} \times \mathcal{J} \rightarrow \mathcal{J}$ denote the product functor.

Theorem (cf. 23.9 and 23.12). (1) If $X, Y \in \text{ob}\mathcal{J}$, then $M(X \mu Y)$ is naturally isomorphic to $M(X) \otimes M(Y)$; and (2) if X has cardinality one, then $M(X)$ is isomorphic to the scalar field.

This last theorem contains the basic ingredients needed to prove that M can be extended to be a "strong", symmetric, monoidal functor (cf. 23.18). From this, it follows that M will transform any algebraic structure that might occur at the level of \mathcal{J} to an analogous algebraic structure at the level of \mathcal{D} . Thus we have

Theorem (cf. 24.24 and 25.10). Let \mathcal{O} and \mathcal{B} be categories such that either (a) $\mathcal{O} = k\text{-spaces}$ and $\mathcal{B} = \mathcal{D}\text{-coalgebras}$; (b) $\mathcal{O} = k\text{-monoids}$ and $\mathcal{B} = \mathcal{D}\text{-bialgebras}$; or (c) $\mathcal{O} = k\text{-groups}$ and $\mathcal{B} = \mathcal{D}\text{-Hopf algebras}$. Then M can be regarded as a functor from \mathcal{O} to \mathcal{B} . Furthermore there exists a functor $\Omega: \mathcal{B} \rightarrow \mathcal{O}$ such that if $G \in \text{ob}\mathcal{O}$ and $H \in \text{ob}\mathcal{B}$, then $\text{Mor}_{\mathcal{O}}(G, \Omega(H))$ is naturally isomorphic to $\text{Mor}_{\mathcal{B}}(M(G), H)$.

Loosely speaking, a \mathcal{D} -algebra is a p -reflexive space together with an associative multiplication and a unit; a \mathcal{D} -coalgebra is a p -reflexive space together with a co-associative comultiplication and a counit; a \mathcal{D} -bialgebra

is a p -reflexive space which simultaneously has the structure of a \mathcal{D} -algebra and a \mathcal{D} -coalgebra with the property that the comultiplication and counit maps are \mathcal{D} -algebra morphisms; and a \mathcal{D} -Hopf algebra is a \mathcal{D} -bialgebra which admits an antipode. k -monoids and k -groups are k -spaces which satisfy the axioms of topological monoids and topological groups respectively, except that the k -space product is used in the axioms in place of the usual product.

Finally, I would like to mention a result only peripherally related to the preceding. A p -reflexive space E will be called a dF space if there exists a Frechet space H such that E is \mathcal{D} -isomorphic to H' , where H' has the topology of precompact convergence. The concept of dF space enables me to prove a generalization of the Banach-Dieudonne theorem.

Theorem (cf. 8.6). Let E be a metrizable locally convex space and F be a dF space. Then the topology of precompact convergence is the finest topology on $\text{hom}(E,F)$ which gives to each equicontinuous, pointwise precompact subset of $\text{hom}(E,F)$ the same relative topology as the topology of pointwise convergence.

Guide to the Reader

1. In this paper, the following conventions will be used:

[14] will refer to item 14 in the bibliography;

15.6 will refer to the sixth item in section 15 of this paper;

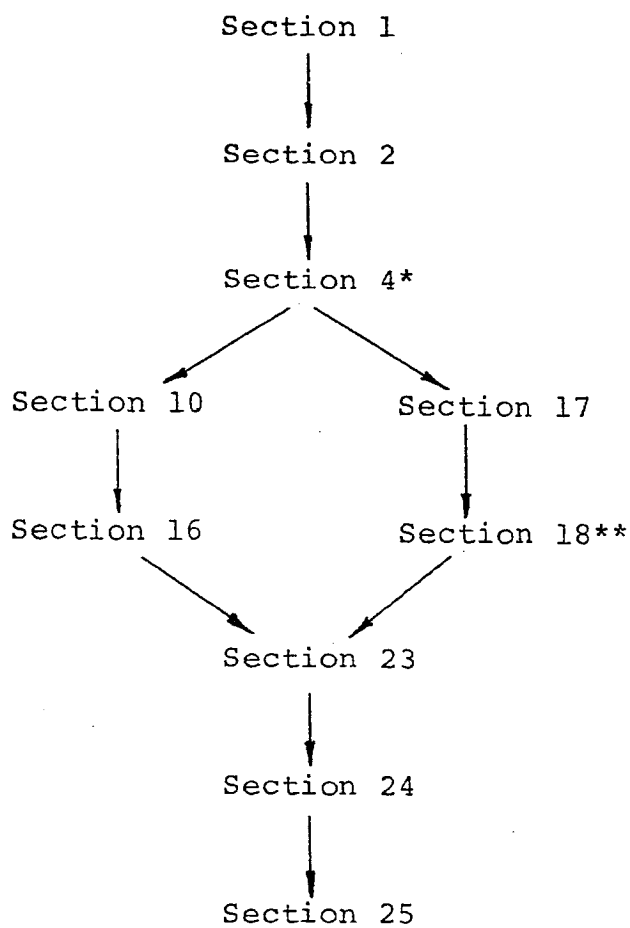
C7 will refer to the seventh item in appendix C.

23.4b will refer to the second numbered item in the proof of 23.4; and

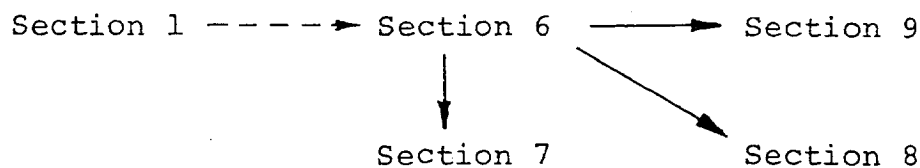
"iff" will mean "if and only if".

2. I have made an attempt to give a rather complete logical development of everything I do. As a result, many of the propositions which I state and prove are trivial and/or well-known. While an attempt is made to attribute important ideas, in many cases I have felt it unnecessary to state that a particular result is trivial and/or well-known. Thus if a proposition has no attribution, it should not be assumed (1) that the result is original with me, or (2) that I attach any value to it.

3. The primary goal of this paper is the group of theorems relating k -groups and \mathcal{D} -Hopf algebras, i.e. theorems 24.24, 25.7, and 25.10. Following is a graph indicating the shortest route to these theorems together with the logical interdependence of the sections which lead to them:

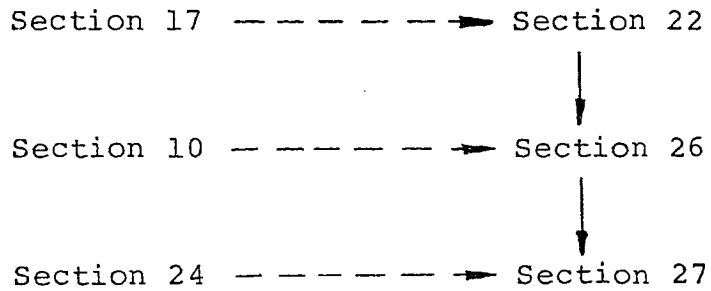


Of secondary interest are the development of dF spaces and the generalization of the Banach-Dieudonne theorem, i.e. sections 6, 7, 8, and 9. Following is a graph indicating the logical interdependence of these sections (the dashed arrows indicate that the section(s) to the left of the arrow is background material for the section to the right):

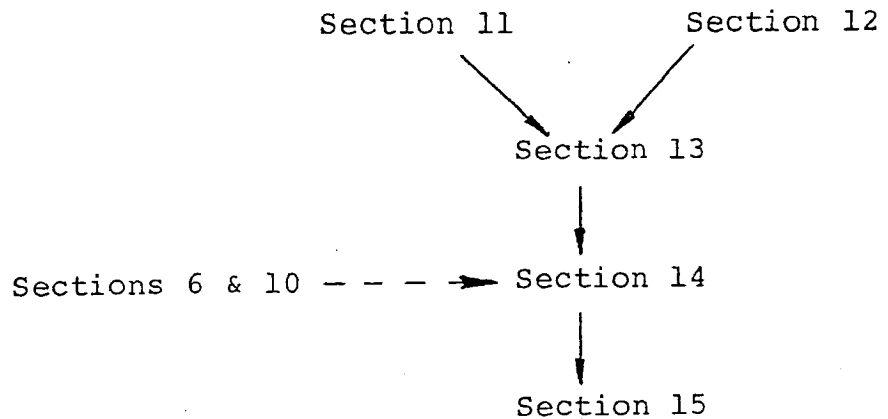
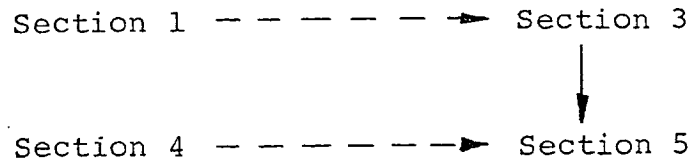


-
- * - Specifically 4.1 through 4.4, and 4.9.
 ** - Specifically 18.1 through 18.3.

Following is the logical interdependence of the sections which lead up to the discussion (in section 27) of representation theory (a dashed arrow will indicate that the section(s) to the left of the arrow is background material for the section to the right):



Following is the logical interdependence of the remaining sections (using the dashed arrow convention mentioned above):



Section 18 -----> Section 19

Sections 8 & 9 -----> Section 20

Sections 10 & 17 -----> Section 21

4. Background material related to category theory can be found in [29], [38] and [39], [51], or [30]; while background material related to the theory of locally convex spaces can be found in [3] and [4], [19], or [27].

5. Happy reading!

Chapter One

p-Reflexive Spaces

For technical reasons, it will be convenient when working with topological vector spaces to restrict ourselves to a quite general class of topological vector space which I ^{call} $I_{\wedge} p$ -reflexive. A space E is p -reflexive provided the evaluation map from E to $(E^p)^p$ is a topological linear isomorphism, where E^p denotes the dual of E with the topology of precompact convergence.

Köthe deals with these and related spaces in §23.9 of [27]. Independent of my studies, Dazord and Jourlin (cf. [10] and [11]) have studied these same concepts more recently and in greater depth.

A locally convex space is p -reflexive if and only if it has two other properties which I call p -determinedness and p -completeness. A space E is p -determined provided the evaluation map from E to $(E^p)^p$ is continuous and is p -complete provided this same map is bijective. p -determinedness is very analogous to quasi-barrelledness and p -completeness is very analogous to semi-reflexivity.

The properties of p -determinedness and p -completeness are studied. In particular, it is shown that p -determined spaces form a complete and cocomplete, full, coreflective subcategory of the category of locally convex spaces and continuous linear maps; and p -complete spaces form a complete and cocomplete, full, reflective subcategory of the category of locally convex Hausdorff spaces and continuous linear maps.

As a consequence of the above information, it is shown that the category of p -reflexive spaces and contin-

uous linear maps is complete and cocomplete and that it is equivalent to its dual category.

Finally a subclass of p -reflexive spaces is introduced which are called dF spaces. dF spaces are "dual" to the class of Frechet spaces in the sense that E is a dF space if and only if E^p is a Frechet space. dF spaces (not to be confused with the DF spaces of Grothendieck) are studied and are shown to share many of the nice properties of Frechet spaces. For example, they are k -spaces and Pták spaces.

A generalization of the Banach-Dieudonne theorem (cf. p.245 of [19]) is proved for the space $\text{hom}_p(E,F)$, where E is metrizable, F is a dF space, and $\text{hom}_p(E,F)$ denotes all continuous linear maps from E to F with the topology of precompact convergence. The classical Banach-Dieudonne theorem is the special case when F equals the scalars.

dF spaces will reappear in chapters two and three.

Section 1 - p-determined, p-complete, & p-reflexive spaces

This section defines p-determined, p-complete, and p-reflexive spaces, and proves that various properties are equivalent to the definitions.

Also in this section we investigate what can be said about spaces of linear maps with regard to the above properties.

1.1 Conventions. Throughout this paper:

Sets will denote the category of sets and functions.

If \mathcal{C} is a category, then \mathcal{C}^{op} will denote the opposite or dual category.

If $F: \mathcal{C} \rightarrow \mathcal{D}$ is a functor, then $F^{\text{op}}: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$ will denote the opposite or dual functor.

All functors will be "covariant". Thus a "contravariant" functor F from \mathcal{A} to \mathcal{B} will be denoted by either $F: \mathcal{A}^{\text{op}} \rightarrow \mathcal{B}$ or $F: \mathcal{A} \rightarrow \mathcal{B}^{\text{op}}$.

If \mathcal{C} is a category, $\text{Mor}_{\mathcal{C}}$ and $\text{ob } \mathcal{C}$ will denote the morphisms and objects of \mathcal{C} respectively, and if $A \in \text{ob } \mathcal{C}$ and $B \in \text{ob } \mathcal{C}$, then $\text{Mor}_{\mathcal{C}}(A, B)$ will denote all morphisms of \mathcal{C} with domain A and codomain B .

\mathbb{K} will denote the scalar field which will be either the complex or real numbers.

\mathcal{C}_n will denote the category of locally convex topological vector spaces over \mathbb{K} and continuous \mathbb{K} -linear

maps.

\mathcal{C} will denote the full subcategory of \mathcal{C}_n consisting of locally convex Hausdorff spaces.

If E and F are locally convex spaces, then $\text{hom}(E, F)$ will denote all continuous linear maps from E to F (note that $\text{hom}(E, F) = \text{Mor}_{\mathcal{C}_n}(E, F)$); and $\text{hom}_p(E, F)$ will denote the vector space $\text{hom}(E, F)$ with the locally convex topology of uniform convergence on precompact sets of E .

If E is a locally convex space, E^p will denote $\text{hom}_p(E, \mathbb{K})$; and if $A \subset E$, A° will denote the polar of the set A (cf. p. 190 of [19]).

Note that $(E, F) \mapsto \text{hom}(E, F)$, $(E, F) \mapsto \text{hom}_p(E, F)$, and $E \mapsto E^p$ are the object maps of functors from $\mathcal{C}_n^{\text{op}} \times \mathcal{C}_n \rightarrow \text{Sets}$, $\mathcal{C}_n^{\text{op}} \times \mathcal{C}_n \rightarrow \mathcal{C}$, and $\mathcal{C} \rightarrow \mathcal{C}$ respectively. The corresponding morphism maps being $(f, h) \mapsto (g \mapsto h \circ g \circ f)$, $(f, h) \mapsto (g \mapsto h \circ g \circ f)$, and $f \mapsto (g \mapsto g \circ f)$ respectively.

Following are two technical lemmas which will be needed in the future.

1.2 Lemma. Let E be a locally convex topological vector space. Suppose $A \subset E$ and for all $x \in E$, $A \cap \{\lambda x : |\lambda| \leq 1\}$ is a neighborhood of zero in $\{\lambda x : |\lambda| \leq 1\}$ for the relative topology induced by E . Then A absorbs points.

Proof: Let $x \in E$. Then there exists a neighborhood of zero $U \subset E$ such that $\{\lambda x : |\lambda| \leq 1\} \cap U \subset \{\lambda x : |\lambda| \leq 1\} \cap A$. So there exists a $\zeta > 0$ such that if $|\lambda| \leq \zeta$, then $\lambda x \in U$. Let $\zeta^* = \inf\{\zeta, 1\}$. Then

if $|\lambda| \leq \zeta^*$, $\lambda x \in \{\lambda x : |\lambda| \leq 1\} \cap U$. So $|\lambda| \leq \zeta^*$ implies $\lambda x \in A$. QED

1.3 Lemma. If E is a topological vector space, then for all $x \in E$, $\{\lambda x : |\lambda| \leq 1\}$ is a compact subset of E .

Proof: The canonical bilinear map from $\mathbb{K} \times E \rightarrow E$ is continuous. Also $\{\lambda \in \mathbb{K} : |\lambda| \leq 1\} \times \{x\}$ is compact in the product space, so the result follows. QED

The following definition shortly will be shown to be equivalent to the property stated in the introduction.

1.4 Definition. A locally convex (not necessarily Hausdorff) topological vector space E will be called p-determined iff a closed, convex, balanced subset A of E is a neighborhood of zero provided that for each precompact subset P of E which contains zero, $A \cap P$ is a neighborhood of zero in P for the relative topology induced by E . Let \mathcal{O}_n and \mathcal{O} denote the full subcategories of \mathcal{C}_n and \mathcal{C} respectively consisting of p-determined spaces.

1.5 Lemma. If E is a p-determined locally convex space, F is any locally convex space, and Q is a set of continuous linear functions from E to F such that for all precompact subsets P of E which contain zero, $\{q|_P : q \in Q\}$ is equicontinuous at zero, then Q is an equicontinuous family.

Proof: Let U be a closed, balanced, convex neighborhood of zero in F . In order to show that Q is equicontinuous, it suffices to show that $\bigcap\{q^{-1}(U) : q \in Q\}$ is a neighborhood of zero in E . But E is assumed to be p -determined. So suppose P is a precompact subset of E containing zero. Then

$$P \cap (\bigcap\{q^{-1}(U) : q \in Q\}) = \bigcap\{(q|_P)^{-1}(U) : q \in Q\}.$$

Now $\{q|_P : q \in Q\}$ as a family of functions from P to F is equicontinuous at zero. So $P \cap \bigcap\{q^{-1}(U) : q \in Q\}$ is a neighborhood of zero in P for the relative topology induced on P by E . Hence we would be done if $\bigcap\{q^{-1}(U) : q \in Q\}$ were closed, convex, and balanced.

But recalling that U is closed, convex, and balanced and that each element of Q is continuous, it readily follows that $\bigcap\{q^{-1}(U) : q \in Q\}$ is closed, convex, and balanced. QED

1.6 Corollary. Let E be a p -determined space and F be any locally convex space. Then if Q is a precompact subset of $\text{hom}_p(E, F)$, then Q is equicontinuous.

Proof: Let P be a precompact subset of E containing zero. By Ascoli's theorem (cf. §2, theorem 2 of [8]) $\{q|_P : q \in Q\}$ is equicontinuous at zero. So the result follows by applying the last lemma. QED

1.7 Proposition. Let E be a locally convex topological vector space. Then the following are equivalent:

- 1) E is p -determined.

2) A closed, convex subset A of E is a neighborhood of zero provided that for each precompact subset P of E which contains zero, $A \cap P$ is a neighborhood of zero in P for the relative topology induced by E .

3) Precompact subsets of E^P are equicontinuous.

4) The evaluation map from E to E^{PP} is continuous.

Proof: The last corollary proves 1) implies 3). In order to prove 3) implies 4), let $r: E \rightarrow E^{PP}$ be the evaluation map. Let U be a neighborhood of zero in E^{PP} . Then there exists a precompact subset P of E^P such that $P^\circ \subset U$. Since we are assuming 3), P is equicontinuous. Hence $\{e \in E : |p(e)| \leq 1 \text{ for all } p \in P\}$ is a neighborhood of zero in E and

$$r(\{e \in E : |p(e)| \leq 1 \text{ for all } p \in P\}) \subset \{\phi \in E^{PP} : |\phi(p)| \leq 1 \forall p \in P\} \subset U.$$

So r is continuous. Hence 3) implies 4).

Now I will show that 4) implies 1). Let A be a closed, convex, balanced subset of E such that for every precompact set P which contains zero, $A \cap P$ is a neighborhood of zero in P .

Claim: For all precompact subsets P of E , $A^\circ|_P$ is uniformly equicontinuous.

Let P be precompact in E . Without loss of generality we may assume that $P \neq \emptyset$. Let $\varepsilon > 0$ be given. Since the map $(x, y) \mapsto (1/\varepsilon)(x - y)$ from $E \times E \rightarrow E$ is uniformly continuous, $(1/\varepsilon)(P - P)$ is precompact. It also contains zero, since $P \neq \emptyset$. Let $P' = (1/\varepsilon)(P - P)$.

By hypothesis there exists a neighborhood of zero, V , so that $V \cap P' \subset A \cap P'$. Let $(x, y) \in \{(r, s) \in P \times P : r - s \in \varepsilon V\}$. Then $(1/\varepsilon)(x - y) \in P' \cap V \subset A$. Suppose $\phi \in A^\circ$. Then $|\phi((1/\varepsilon)(x - y))| \leq 1$. So $|\phi(x) - \phi(y)| \leq \varepsilon$. So $A^\circ|P$ is uniformly equicontinuous. QED on claim

Claim: For all $e \in E$, $\{\phi(e) : \phi \in A^\circ\}$ is precompact.

Let $e \in E$. Now A absorbs e by 1.2 and 1.3. So there exists a $\lambda > 0$ such that $\lambda e \in A$. But then $|\phi(\lambda e)| \leq 1$ for all $\phi \in A^\circ$. So $|\phi(e)| \leq 1/\lambda$, for all $\phi \in A^\circ$. Hence $\{\phi(e) : \phi \in A^\circ\}$ is precompact.

QED on claim

Claim: A° is a precompact subset of E^P .

This follows from the last two claims by Ascoli's theorem (§2, theorem 2 of [8]). QED on claim

So $\{\phi \in E^{PP} : |\phi(r)| \leq 1 \text{ for all } r \in A^\circ\}$ is a neighborhood of zero in E^{PP} . Let $r: E \rightarrow E^{PP}$ denote the evaluation map which by assumption is continuous. Thus $r^{-1}(\{\phi \in E^{PP} : |\phi(r)| \leq 1 \text{ for all } r \in A^\circ\}) = A^{\circ\circ}$ is a neighborhood of zero. Now $A \neq \emptyset$, since it is absorbing. So by the bipolar theorem $A^{\circ\circ} = A$, since by assumption A is closed, balanced, and convex. Hence A is a neighborhood of zero. Hence we have shown that 4) implies 1).

Obviously 2) implies 1). I will now complete the chain of implications by showing that 1) implies 2).

Suppose A is closed, convex, and has the property stated in 2). I would like to conclude that A is a

neighborhood of zero. Let B be the balanced core of A , i.e. the largest balanced set contained in A . Either $B = \emptyset$ or $B = \cap \{\lambda A : |\lambda| \geq 1\}$ (cf. [19] p. 80). So B is closed, balanced, and convex. Let P be any precompact set containing zero. Let P' be the balanced hull of P . $P' = \{\lambda p : |\lambda| \leq 1 \text{ and } p \in P\}$ is precompact. So there exists a balanced neighborhood of zero V such that $V \cap P' \subset A \cap P' \subset A$. V is balanced as is P' , hence $V \cap P'$ is balanced. So $V \cap P' \subset B$. So $V \cap P \subset B \cap P$. So $B \cap P$ is a neighborhood of zero in P with the relative topology. This is true for all P precompact containing zero. Hence since E is assumed to be p -determined, B is a neighborhood of zero. But $B \subset A$. So A is a neighborhood of zero.

QED

1.8 Remark. The third statement of the last theorem shows that barrelled spaces are p -determined. The following proposition shows that metrizable spaces are also p -determined.

1.9 Proposition. A locally convex metrizable space is p -determined.

Proof: Let E be locally convex and metrizable. Suppose $A \subset E^P$ is precompact, then $A|P: P \rightarrow K$ is equicontinuous, for all P precompact in E , by Ascoli. Let $\{U_n : n \in \omega\}$ be a countable base for the neighborhood system at zero such that $n \geq m \Rightarrow U_n \subset U_m$. Suppose A is not equicontinuous at zero. Then there exists an

$\varepsilon > 0$ such that for all $n \in \omega$, $A(U_n) \not\subset \{k \in \mathbb{K} : |k| \leq \varepsilon\}$. For all $n \in \omega$, choose $x_n \in U_n$ such that there exists a $p \in A$ such that $|p(x_n)| > \varepsilon$. Now $x_n \rightarrow 0$ and a convergent sequence together with its limit are compact. Hence $P = \{0\} \cup \{x_n : n \in \omega\}$ is precompact. But $A|_P$ is equicontinuous at zero and $x_n \rightarrow 0$ in P . So there exists N such that $n \geq N$ and $p \in A$ implies that $|p(x_n)| \leq \varepsilon$. Contradiction. So A is equicontinuous at zero and thus is equicontinuous. QED

The following propositions illustrate some of the permanence properties of p -determined spaces. Other permanence properties will be discussed in section four.

1.10 Proposition. Let I be an index set and V be a vector space. Suppose for all $\alpha \in I$, E_α is a p -determined space and $f_\alpha: E_\alpha \rightarrow V$ is a linear map. Let \mathcal{J} be the finest locally convex topology on V which makes each f_α continuous. Then (V, \mathcal{J}) is a p -determined space.

Note: the case where I equals the null set is not ruled out.

Proof: Suppose A is a convex, balanced, \mathcal{J} -closed subset of V such that $A \cap P$ is a neighborhood of zero in P for the relative topology induced on P by (V, \mathcal{J}) , for all P which are precompact in (V, \mathcal{J}) and which contain zero. By 1.2 and 1.3, A absorbs points. So A will be a \mathcal{J} -neighborhood of zero provided $f_\alpha^{-1}(A)$ is a

neighborhood of zero for all $\alpha \in I$.

Let $\alpha \in I$, then $f_\alpha^{-1}(A)$ is closed, balanced, and convex. Let P be a precompact subset of E_α which contains zero. Then $f_\alpha(P)$ is precompact in (V, \mathcal{J}) and contains zero. So there exists a neighborhood U of zero in (V, \mathcal{J}) such that $U \cap f_\alpha(P) \subset A \cap f_\alpha(P)$. This implies that $f_\alpha^{-1}(U) \cap P \subset f_\alpha^{-1}(A) \cap P$. But $f_\alpha^{-1}(U)$ is a neighborhood of zero. So $f_\alpha^{-1}(A) \cap P$ is a neighborhood of zero in P . This is true for all P precompact which contain zero. Hence $f_\alpha^{-1}(A)$ is a neighborhood of zero in E_α , because E_α is p -determined.

So A is a (V, \mathcal{J}) neighborhood of zero. Hence (V, \mathcal{J}) is p -determined. QED

1.11 Proposition.

1) The locally convex direct sum of p -determined spaces is p -determined.

2) If E is a p -determined space and A is a linear subspace of E (not necessarily closed), then E/A is p -determined.

3) The category \mathcal{O}_n of p -determined spaces is a cocomplete category as is the category \mathcal{O} of Hausdorff p -determined spaces.

4) The \mathcal{O}_n -colimit of a diagram in \mathcal{O}_n agrees with the \mathcal{C}_n -colimit of that same diagram.

5) The \mathcal{O} -colimit of a diagram in \mathcal{O} agrees with the \mathcal{C} -colimit of that same diagram.

Proof: 1) and 2) follow immediately from 1.10. By 8.4.2 of [38], \mathcal{O}_n and \mathcal{O} are cocomplete if coproducts and difference cokernels exist in them. 1) and 2) guarantee that they do, since for example if $f: A \rightarrow B$ is a \mathcal{O} -morphism (respectively \mathcal{O}_n -morphism), then the canonical projection from B to $B/f(A)^\sim$ (respectively, from B to $B/f(A)$) is the cokernal of f . Since the inclusion functor from \mathcal{O} (respectively, \mathcal{O}_n) to \mathcal{C} (respectively, \mathcal{C}_n) preserves coproducts and difference cokernels; by 7.4.5 of [38], it preserves colimits. Thus 4) and 5) follow. QED

1.12 Definition. A locally convex topological vector space E will be called p-complete provided it is Hausdorff and that closed, precompact subsets of E are complete. Let \mathcal{B} denote the full subcategory of \mathcal{C} consisting of p-complete spaces.

1.13 Proposition. Let E be a locally convex Hausdorff space. Then the following are equivalent:

- 1) E is p-complete.
- 2) The evaluation map from E to E^{pp} is a surjection.

Proof: That 1) implies 2) is an easy consequence of Mackey's theorem ([19] p. 205). The reasoning is as follows. Let P be a precompact subset of E . Let P' equal the closed balanced convex hull of P . P' is also precompact. So P' is complete. Hence P' is compact,

convex, and balanced. So P' is $\sigma(E, E')$ -compact, convex, and balanced. This says that the topology of E^P is coarser than $\tau(E', E)$ and finer than $\sigma(E', E)$, so by Mackey's theorem $(E^P)' = E$, i.e. the evaluation map is surjective.

To show that 2) implies 1), note that 2) implies that the topology of E^P on E' is compatible with the dual pair (E', E) . So Mackey's theorem tells us that there is a collection, \mathcal{G} , of $\sigma(E, E')$ -compact, convex, balanced subsets of E such that $\cup \mathcal{G} = E$ and that the topology of precompact convergence equals the topology of uniform convergence on the sets of \mathcal{G} , as topologies on E' .

Suppose P is closed and precompact in E . Then P° is a neighborhood of zero in E^P . Hence there exists $\{S_1, \dots, S_n\} \subset \mathcal{G}$ and $\varepsilon > 0$ such that

$$\varepsilon\{\phi \in E' : |\phi(x)| \leq 1 \text{ for all } x \in \cup_{i=1}^n S_i\} =$$

$$\{\phi \in E' : |\phi(x)| \leq \varepsilon \text{ for all } x \in \cup_{i=1}^n S_i\} \subset P^\circ,$$

i.e. $\varepsilon(\cup_{i=1}^n S_i)^\circ \subset P^\circ$. But then $P \subset P^\circ \subset (1/\varepsilon)(\cup_{i=1}^n S_i)^\circ$.

Now $(\cup_{i=1}^n S_i)^\circ$ is the $\sigma(E, E')$ -closed, convex, balanced hull of $\cup_{i=1}^n S_i$, by the bipolar theorem. $(\cup_{i=1}^n S_i)^\circ$ is thus $\sigma(E, E')$ -compact by proposition 2 of §3.5 of [19].

So $(\cup_{i=1}^n S_i)^\circ$ is $\sigma(E, E')$ -complete.

Claim: $(\cup_{i=1}^n S_i)^\circ$ is complete for the uniformity of E^P .

Let $\Gamma_1 = \{\{\phi\} : \phi \in E'\}$ and $\Gamma_2 =$ all equicontinuous subsets of E' . Note that $\cup \Gamma_1 = E' = \cup \Gamma_2$ and $\Gamma_1 \subset \Gamma_2$. $(\cup_{i=1}^n S_i)^\circ$ with the relative $\sigma(E, E')$ -uniform-

ity is just Γ_1 convergence when considered in $\mathbb{K}^{E'}$ and the E^P uniformity is just Γ_2 convergence when considered in $\mathbb{K}^{E'}$. So by corollary 2 to Proposition 5, §1 of [8], $(\cup_{i=1}^n S_i)^{\circ\circ}$ is complete for the E^P uniformity, since it is complete for the coarser uniformity.

QED on claim

P is a closed subset of the complete space

$(1/\varepsilon)(\cup_{i=1}^n S_i)^{\circ\circ}$, hence P is complete in E^P . Hence

E is p -complete.

QED

1.14 Note. A locally convex Hausdorff space is quasi-complete if it is complete, and is p -complete if it is quasi-complete.

Following are propositions which illustrate some of the permanence properties of p -complete spaces. Other permanence properties can be found in section four.

1.15 Proposition. The product in \mathcal{C} of a collection $\{E_i : i \in I\}$ of p -complete spaces is p -complete.

Proof: Let $E = \prod\{E_i : i \in I\}$. Let A be a closed precompact subset of E . For all $i \in I$, let $\pi_i : E \rightarrow E_i$ denote the canonical projection. For all $i \in I$, let $B_i = \pi_i(A)^-$. Each B_i is a closed, precompact subset of E_i and is therefore complete. Let $B = \prod\{B_i : i \in I\}$. Since the product of complete spaces is complete, B is complete. For all $i \in I$, let $p_i : B \rightarrow B_i$ be the canonical projection. For all $i \in I$, let $f_i : B_i \rightarrow E_i$ be the inclusion map. Using the universal property of pro-

ducts, let $f: B \rightarrow E$ be the unique uniformly continuous map such that for all $i \in I$,

$$\begin{array}{ccc}
 & f & \\
 B & \longrightarrow & E \\
 \pi_i \downarrow & & \downarrow \pi_i \\
 B_i & \xrightarrow{f_i} & E_i
 \end{array} \quad \text{commutes.}$$

Define for each $i \in I$, a uniformly continuous map $s_i: A \rightarrow B_i$ by $s_i(a) = \pi_i(a)$. Then there exists a unique uniformly continuous map $g: A \rightarrow B$ such that

$$\begin{array}{ccc}
 A & \xrightarrow{g} & B \\
 & \searrow s_i & \downarrow \pi_i \\
 & & B_i
 \end{array} \quad \text{commutes for all } i \in I.$$

Hence combining the last two diagrams, we find that

$$\begin{array}{ccc}
 A & \xrightarrow{f \circ g} & E \\
 s_i \downarrow & & \downarrow \pi_i \\
 B_i & \xrightarrow{f_i} & E_i
 \end{array} \quad \text{commutes.}$$

Now let $h: A \rightarrow E$ be the inclusion map. It is then easy to check from the definitions that

$$\begin{array}{ccc}
 A & \xrightarrow{h} & E \\
 s_i \downarrow & & \downarrow \pi_i \\
 B_i & \xrightarrow{f_i} & E_i
 \end{array} \quad \text{commutes.} \quad \text{But by the}$$

universal property of products, there is at most one such map, hence $h = f \circ g$, i.e.

$$\begin{array}{ccc}
 A & \xrightarrow{h} & E \\
 & \searrow g & \nearrow f \\
 & & B
 \end{array} \quad \text{commutes.}$$

This factorization is what we need in order to complete

the proof.

Suppose x_α is a cauchy net in A , then $g(x_\alpha)$ is a cauchy net in B . Since B is complete, there is a $b \in B$ such that $g(x_\alpha) \longrightarrow b$. Hence

$x_\alpha = h(x_\alpha) = f(g(x_\alpha)) \longrightarrow f(b)$ in E . But A is closed in E , so $f(b) \in A$. Hence A is complete. Hence E is p -complete. QED

1.16 Proposition. Closed linear subspaces of p -complete spaces are p -complete.

Proof: Let E be a p -complete space and let A be a closed linear subspace of E . Let R be a closed precompact subspace of A . Since A is closed, R is a closed and precompact in E . Hence it is complete. QED

1.17 Corollary. The category \mathcal{B} of p -complete spaces is complete. Moreover the \mathcal{B} -limit of a diagram in \mathcal{B} agrees with the \mathcal{C} -limit of that same diagram.

Proof: By 7.4.2 of [38], \mathcal{B} is complete iff \mathcal{B} has products and difference kernels. Products exist in \mathcal{B} by 1.15. If A and B are p -complete and $f: A \longrightarrow B$ is continuous and linear, then $f^{-1}(\{0\})$ is a p -complete space by 1.16. Hence $\text{Kernel}(f) = (\text{the inclusion map of } f^{-1}(\{0\}) \text{ into } A)$ is also the \mathcal{B} -kernel. So difference kernels exist in \mathcal{B} . Hence \mathcal{B} is complete. Since the inclusion functor of \mathcal{B} into \mathcal{C} preserves products and difference kernels, by 7.4.5 of [38], it preserves limits. This says that the limits in \mathcal{B} agree with those in \mathcal{C} . QED

The following are technical lemmas which will be needed in our discussion of p -reflexive spaces.

1.18 Lemma. Let E be a locally convex space and $e: E \rightarrow E^{PP}$ be the evaluation map. If x_α is a net in E such that $e(x_\alpha) \rightarrow 0$, then $x_\alpha \rightarrow 0$.

Proof: Suppose $e(x_\alpha) \rightarrow 0$. Let U be a closed, convex, balanced subset of E . Then U° is an equicontinuous subset of E' and U° is $\sigma(E', E)$ bounded. So by Ascoli's theorem U° is precompact in E^P . Hence $\{\phi \in E^{PP} : |\phi(f)| \leq 1 \text{ for all } f \in U^\circ\}$ is a neighborhood of zero in E^{PP} . So $e(x_\alpha)$ is eventually in it. Thus x_α is eventually in $U^{\circ\circ}$. But since U is closed, convex, and balanced, $U^{\circ\circ} = U$. Hence x_α is eventually in U . So $x_\alpha \rightarrow 0$. QED

1.19 Lemma. Let E be a locally convex space and $e: E \rightarrow E^{PP}$ be the evaluation map. Then e is injective iff E is Hausdorff.

Proof: Suppose E is Hausdorff. Suppose $x \in E$ and $x \neq 0$. Then since $\{0\}$ is a closed linear subspace of E and $x \notin \{0\}$, there exists a continuous linear functional ϕ such that $\phi(x) \neq 0$. But then $[e(x)](\phi) \neq 0$. So $e(x) \neq 0$. Hence e is injective.

Suppose e is injective. Suppose $x \in E$ and $x \neq 0$. Then $e(x) \neq 0$. So there exist a continuous linear functional ϕ such that $[e(x)](\phi) \neq 0$. Hence $\phi(x) \neq 0$. Then $\{y \in E : |\phi(y)| < |\phi(x)|\}$ is a neighborhood of

zero which does not contain x . Hence E is Hausdorff.

QED

1.20 Definition. A locally convex space E is p-reflexive iff the canonical map from E to E^{pp} is a topological linear isomorphism. Let \mathcal{D} denote the full subcategory of \mathcal{C}_n consisting of p-reflexive spaces.

1.21 Theorem. Let E be a locally convex space. Then E is p-reflexive iff E is both p-complete and p-determined.

Proof: Suppose E is p-reflexive, then E is Hausdorff by 1.19, since isomorphisms are injective; and E is p-complete and p-determined, since the canonical map from E to E^{pp} is both surjective and continuous (cf. 1.7 and 1.13).

Suppose E is p-complete and p-determined. Let e be the canonical map from E to E^{pp} . e is injective by the last lemma, since E is Hausdorff. It is surjective and continuous since E is p-complete and p-determined. So the proof will be completed if it can be shown that $e(x_\alpha) \rightarrow 0$ implies $x_\alpha \rightarrow 0$. But this is true by 1.18. QED

1.22 Remark. This shows that the concept of p-reflexivity encompasses a great many locally convex spaces. In particular, all spaces which are barrelled and quasi-complete are p-reflexive, for example all Frechet spaces are p-reflexive.

The following will lead into the study of "duals" of p -reflexive spaces.

1.23 Proposition. Let E be a p -determined space and F be a p -complete space. Then $\text{hom}_p(E, F)$ is p -complete.

Proof: $\text{hom}_p(E, F)$ is Hausdorff by prop. 2 of [4], Chap. 3, §3.

Let H be a closed, precompact subset of $\text{hom}_p(E, F)$. By 1.6, H is an equicontinuous family of functions. For all $x \in E$, define $\phi_x: \text{hom}_p(E, F) \rightarrow F$ by $\phi_x(f) = f(x)$. Each ϕ_x is uniformly continuous. Therefore for all $x \in E$, $H(x) = \phi_x(H)$ is a precompact subset of F . Hence $H(x)^-$ is closed and precompact and therefore complete, since F is p -complete.

Let f_α be a cauchy net in H . Let $x \in E$. $\phi_x(f_\alpha) = f_\alpha(x)$ is thus a cauchy net in $H(x)^-$, so it converges. Let $u(x) = \lim_{\alpha} f_\alpha(x)$. If we repeat this process for each $x \in E$, we define a function $u: E \rightarrow F$, and f_α converges to u in the topology of simple convergence.

Now by the corollary to prop. 5 of [4], Chap. 3, §3, u is a continuous linear function from E to F and f_α converges to u in $\text{hom}_p(E, F)$. But H is closed, so $u \in H$.

Hence we have shown that every cauchy net in H converges to an element of H . Hence H is complete. So $\text{hom}_p(E, F)$ is p -complete. QED

1.24 Corollary. If E is a p -determined space, then

E^P is p -complete.

Proof: $E^P = \text{hom}_p(E, K)$. Now apply 1.23. QED

1.25 Proposition. If E is p -complete, then E^P is p -determined.

Proof: By 1.7, E^P is p -determined if every precompact subset of E^{PP} is equicontinuous. Let e be the evaluation map from E to E^{PP} . Since E is Hausdorff, e is injective. Since E is p -complete, e is surjective. So by 1.18, $e^{-1}: E^{PP} \rightarrow E$ is a continuous linear map.

Let P be a precompact subset of E^{PP} . Thus $e^{-1}(P)$ is a precompact subset of E . Hence $e^{-1}(P)^\circ$ is a neighborhood of zero in E^P and $e^{-1}(P)^\circ{}^\delta$ is an equicontinuous subset of $(E^P)'$.

Claim: $P \subset e^{-1}(P)^\circ{}^\delta$.

Suppose $p \in P$. Let $\phi \in e^{-1}(P)^\circ$. Thus for all $p \in P$, $|\phi(e^{-1}(p))| \leq 1$. But $p(\phi) = [e(e^{-1}(p))](\phi) = \phi(e^{-1}(p))$. So $|p(\phi)| \leq 1$. So $p \in (e^{-1}(P)^\circ)^\delta$.

Hence $P \subset e^{-1}(P)^\circ{}^\delta$ QED on claim

Thus P is equicontinuous. Hence E^P is p -determined. QED

1.26 Corollary. If a locally convex space E is p -reflexive, then so is E^P .

Proof: 1.21.

Section 2 - Categorical properties of p-reflexive spaces

We prove in this section that the category of p-determined spaces is a coreflective subcategory of the category of all locally convex spaces; that the category of p-complete spaces is a reflective subcategory

of the category of Hausdorff locally convex spaces; that the category of p-reflexive spaces is a reflective subcategory of the category of Hausdorff p-determined spaces; and that the category of p-reflexive spaces is a coreflective subcategory of the category of p-complete spaces.

2.1 Theorem. Let E be a locally convex topological vector space. Let V be the underlying vector space of E . Then there exists a locally convex topology \mathcal{J} on V such that

- 1) \mathcal{J} is finer than the given topology on E ,
- 2) (V, \mathcal{J}) is p-determined, and
- 3) If F is a locally convex p-determined space and $g: F \rightarrow E$ is a continuous linear map, then $g: F \rightarrow (V, \mathcal{J})$ is actually continuous.

Moreover, if P is a precompact subset of E , then the relative uniformities induced on P by E and by (V, \mathcal{J}) agree.

2.2 Corollary. \mathcal{O}_n is a bicoreflective subcategory of \mathcal{C}_n .

Note: For a discussion of bicoreflectivity, see [18].

Proof of 2.2: By B4 and B6, epimorphisms in \mathcal{C}_n are surjections and monomorphisms are injections. Hence the fact that \mathcal{O}_n is bicoreflective in \mathcal{C}_n summarizes 2.1 except it omits the "moreover" part. QED

Proof of 2.1: Let I = the set of all p -determined locally convex topologies on V which are finer than the given topology. For all $\alpha \in I$, let $f_\alpha: (V, \alpha) \rightarrow V$ be defined by $f_\alpha(x) = x$ for all $x \in V$. Let \mathcal{J} be the finest locally convex topology on V which makes each f_α continuous. Notice that \mathcal{J} is finer than the original topology on E , since $f_\alpha: (V, \alpha) \rightarrow E$ is continuous for all $\alpha \in I$.

By 1.10, (V, \mathcal{J}) is p -determined and hence $\mathcal{J} \in I$. Thus \mathcal{J} satisfies 1) and 2) of the conclusion. Now suppose that F is a p -determined space and $g: F \rightarrow E$ is a linear continuous map. Let \mathcal{J}^* denote the finest locally convex topology on V making $g: F \rightarrow V$ continuous. By 1.10, \mathcal{J}^* is a p -determined topology on V ; and also \mathcal{J}^* is finer than the given topology on V , since $g: F \rightarrow E$ is continuous. Hence $\mathcal{J}^* \in I$. So $\mathcal{J} \subset \mathcal{J}^*$ and $f_{\mathcal{J}^*}: (V, \mathcal{J}^*) \rightarrow (V, \mathcal{J})$ is continuous. But (V, \mathcal{J}^*) is a locally convex topology making g continuous. So since $\mathcal{J} \subset \mathcal{J}^*$, $g: F \rightarrow (V, \mathcal{J})$ is continuous. Hence 3) of the conclusion is satisfied.

Now to show that on precompact subset of E , the relative uniformities induced on P by E and by (V, \mathcal{J}) agree. This will require a little work. We will first need a lemma.

2.1a Lemma. Let E and V be as in the statement of 2.1. Let Γ be the set of all $A \subset E$ such that A is convex, balanced and such that $A \cap P$ is a neighborhood of zero in P for the relative topology induced by E , for all $P \subset E$ which are precompact in E and which contain zero. Then Γ is a base of the neighborhood system at zero for a locally convex topology on the underlying vector space V .

Proof of lemma: By 1.2 and 1.3 each element of Γ is absorbing.

Claim: If $A, B \in \Gamma$, then $A \cap B \in \Gamma$.

$A \cap B$ is convex and balanced. Suppose P is a precompact subset of E which contains zero. Now there exist neighborhoods U and V of zero such that $U \cap P \subset A \cap P$ and $V \cap P \subset B \cap P$. So $(U \cap V) \cap P \subset (A \cap B) \cap P$. So $A \cap B$ is a neighborhood of zero in P . Thus $A \cap B \in \Gamma$. QED on claim

Claim: If $A \in \Gamma$ and $\lambda > 0$, then $\lambda A \in \Gamma$.

λA is convex and balanced. Suppose P is a precompact subset of E which contains zero. Then the same is true of $(1/\lambda)P$. Since $A \in \Gamma$, there exists a neighborhood of zero U such that $U \cap \lambda^{-1}P \subset A \cap \lambda^{-1}P$. But then $\lambda U \cap P \subset \lambda A \cap P$. So since λU is a neighborhood of zero, $\lambda A \cap P$ is a neighborhood of zero in P .

Hence $\lambda A \in \Gamma$.

QED on claim

So by prop. 6 of §2.4 of [19], Γ is a base for the neighborhood system at zero for a locally convex topology.

QED on lemma

2.1b Lemma. Let \mathcal{J}' be the locally convex topology for which Γ of the last lemma is a base for the neighborhood system at zero. Then

- 1) \mathcal{J}' is finer than the given topology on E ;
- 2) If P is precompact in E , then the relative uniformities on P induced by the topology of E and by \mathcal{J}' agree; and
- 3) (V, \mathcal{J}') is a p -determined space.

Proof of lemma: 1) The set of balanced, convex neighborhoods of zero form a base for the neighborhood system at zero for E ; and each such set is in Γ . Hence \mathcal{J}' is finer than the given topology. QED on 1)

2) Let P be a precompact set in E . Without loss of generality, we may assume that $P \neq \emptyset$. The map from $E \times E \rightarrow E$ defined by $(x, y) \mapsto x - y$ is uniformly continuous, hence the image under this map of $P \times P$, i.e. $P - P$, is precompact. Also $P \neq \emptyset$ implies that $0 \in P - P$. Let R be a neighborhood of zero in \mathcal{J}' . To show that the uniformity induced on P by \mathcal{J}' is coarser than that induced by E , we must find a neighborhood of zero, U , in E such that

$$\{(x, y) \in P \times P : x - y \in U\} \subset \{(x, y) \in P \times P : x - y \in R\}.$$

Let $S \in \Gamma$ such that $S \subset R$. Since $S \in \Gamma$, there exists a neighborhood U of zero in E such that

$U \cap (P - P) \subset S \cap (P - P)$. But this implies that $\{(x,y) \in P \times P : x - y \in U\} \subset \{(x,y) \in P \times P : x - y \in SCR\}$. So the uniformity of P induced by E is finer than the uniformity on P induced by \mathcal{J}' .

But the other inclusion follows since the topology \mathcal{J}' is finer than that of E by 1). So the two relative uniformities are identical. QED on 2)

3) Suppose R is a \mathcal{J}' -closed, convex, balanced set such that $R \cap P$ is a neighborhood of zero in P for the relative topology induced on P by \mathcal{J}' , for every \mathcal{J}' -precompact subset P of V which contains zero. Let L be a precompact subset of E which contains zero. By 2), L is a precompact subset of (V, \mathcal{J}') . So $R \cap L$ is a neighborhood of zero in L for the relative topology on L induced by \mathcal{J}' . By the corollary to prop. 4, §2 of Chap. 2 of [6], the relative topology of the uniform topology is the uniform topology of the relative uniformity. Hence the relative topology on L induced by \mathcal{J}' is the uniform topology of the relative uniformity on L induced by \mathcal{J}' . But by 2) this is just the uniform topology of the relative uniformity on L induced by E . This is (again using the reference to [6] above) just the relative topology on L induced by E . So $R \cap L$ is a neighborhood of zero in L for the relative topology on L induced by E . Hence $R \in \Gamma$ since R is convex and balanced. So R is a \mathcal{J}' -neighborhood of zero. This implies that (V, \mathcal{J}') is p -determined. QED on 3)

QED on lemma

Now we are ready to finish up the proof of 2.1.

Let \mathcal{J}_0 denote the topology of E . By 1) we know that $\mathcal{J}_0 \subset \mathcal{J}$ and 2.1b we know that $\mathcal{J}_0 \subset \mathcal{J}'$. Let \mathcal{U}_0 , \mathcal{U} , and \mathcal{U}' denote the uniformities on V induced by \mathcal{J}_0 , \mathcal{J} , and \mathcal{J}' respectively. Let $q: V \rightarrow V$ be defined by $q(x) = x$. q is a continuous linear map from (V, \mathcal{J}') to E . So since (V, \mathcal{J}') is p -determined by using 3) of 2.1, q is actually continuous from (V, \mathcal{J}') to (V, \mathcal{J}) . Hence $\mathcal{J}' \supset \mathcal{J}$. So $\mathcal{J}' \supset \mathcal{J} \supset \mathcal{J}_0$ and thus $\mathcal{U}' \supset \mathcal{U} \supset \mathcal{U}_0$. Let P be a \mathcal{U}_0 precompact set. Then $\mathcal{U}'|_P \supset \mathcal{U}|_P \supset \mathcal{U}_0|_P$. But $\mathcal{U}_0|_P \supset \mathcal{U}'|_P$ by 2.1b. Hence $\mathcal{U}|_P = \mathcal{U}_0|_P$. So the "moreover" part of 2.1 is proved. QED

2.3 Note. In the last several lines of the above proof I note that $\mathcal{J}' \supset \mathcal{J} \supset \mathcal{J}_0$. In general \mathcal{J}' need not equal \mathcal{J} , and \mathcal{J} need not equal \mathcal{J}_0 .

Proof of note: In [10] there appears an example where \mathcal{J}_0 is properly contained in \mathcal{J} . As for \mathcal{J}' and \mathcal{J} , let F be a non-complete Montel space (for their existence see [26]). Let E be the strong dual of F . E is a Montel space, hence barrelled, hence p -determined. So if we define \mathcal{J} and \mathcal{J}' relative to E , we will find that \mathcal{J} is properly contained in \mathcal{J}' by the following reasoning.

If $\mathcal{J} = \mathcal{J}'$, then E^p will be complete as is easily verified. But E^p equals the strong dual of E and is thus isomorphic to F , since F is a Montel space and is hence reflexive. So if $\mathcal{J} = \mathcal{J}'$, F is complete.

Contradiction. Hence \mathcal{J} is properly contained in \mathcal{J}' .

2.4 Remark. My goal is theorem 2.10. But that theorem is true in the more general setting of Hausdorff uniform spaces and uniformly continuous maps. So let us extend the concept of a p-complete space as below. However whenever I speak of simply a "p-complete space", I will mean a p-complete locally convex Hausdorff space.

2.5 Definition. A uniform space X will be said to be p-complete provided it is Hausdorff and provided every closed, precompact subset of X is complete.

2.6 Theorem. If X is a Hausdorff uniform space, then there exists a p-complete uniform space X^\sim and a uniformly continuous map $i: X \rightarrow X^\sim$ such that if Y is a p-complete uniform space and $f: X \rightarrow Y$ is uniformly continuous, then there exists a unique uniformly continuous map $f^\sim: X^\sim \rightarrow Y$ such that

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 i \downarrow & \nearrow f^\sim & \\
 X^\sim & &
 \end{array}
 \quad \text{commutes.}$$

Moreover 1) i is an isomorphism onto its range, and 2) the range of i is dense in X^\sim .

Proof: First we need quite a few preliminaries.

2.6a Definition. If Z is a Hausdorff uniform space and $A \subset Z$, then A will be called p-closed in Z provided A contains the closure of each of its precompact sets.

2.6b Lemma. If Z is a Hausdorff uniform space, then $\{A \subset Z : A \text{ is } p\text{-closed in } Z\}$ form the closed sets for a topology finer than the uniform topology.

Proof of lemma: a) Suppose \mathcal{O} is a family of p -closed sets. Suppose P is precompact in $\cap \mathcal{O}$. Suppose P^- denotes the Z -closure of P . Since $P \subset \cap \mathcal{O} \subset A$ for all $A \in \mathcal{O}$ and since each $A \in \mathcal{O}$ is p -closed, $P^- \subset A$ for all $A \in \mathcal{O}$. So $P^- \subset \cap \mathcal{O}$. Hence $\cap \mathcal{O}$ is p -closed. b) Suppose \mathcal{O} is a finite family of p -closed sets. Suppose P is precompact and $P \subset \cup \mathcal{O}$. Let $P_A = A \cap P$ for all $A \in \mathcal{O}$. Then for all $A \in \mathcal{O}$, $P_A \subset A$ and each P_A is precompact. Hence $P_A^- \subset A$ for all $A \in \mathcal{O}$. Also $P \subset \cup \{P_A^- : A \in \mathcal{O}\}$. But $\cup \{P_A^- : A \in \mathcal{O}\}$ is closed, so $P^- \subset \cup \{P_A^- : A \in \mathcal{O}\} \subset \cup \mathcal{O}$. Hence $\cup \mathcal{O}$ is p -closed. c) Trivially \emptyset and Z are p -closed.

Hence the p -closed sets form the closed sets of a topology. Suppose $A \subset Z$ and A is closed for the uniform topology. Let $P \subset A$ be precompact. Then $P^- \subset A^- \subset A$. Hence A is p -closed. So the new topology is finer than the uniform topology. QED on lemma

2.6c Definition. Call this new topology the p -topology induced by Z .

2.6d Definition. If Z is a Hausdorff uniform space and $A \subset Z$, let A^\sim denote the closure of A in the p -topology induced by Z . Call A^\sim the p -closure of A .

2.6e Lemma. If R and S are two Hausdorff uniform spaces and if $A: R \rightarrow S$ is uniformly continuous, then A is continuous from the p -topology on R to the p -top-

ology on S .

Proof of lemma: Let M be p -closed in S . It suffices to show that $A^{-1}(M)$ is p -closed in R . Suppose $P \subset A^{-1}(M)$ is precompact. Then $A(P)$ is precompact in S . Also $A(P) \subset M$. So $A(P)^- \subset M$ since M is p -closed. Now A is continuous if both R and S are given the uniform topologies. So $A(P^-) \subset A(P)^-$. Hence $A(P^-) \subset M$, i.e. $P^- \subset A^{-1}(M)$. So $A^{-1}(M)$ is p -closed in R .

QED on lemma

2.6f Lemma. If Z is a Hausdorff uniform space and $A \subset Z$, then

1) if A is p -complete, then A is p -closed in Z ; and 2) if Z is p -complete and A is p -closed in Z , then A is p -complete.

Proof of lemma: 1) Let $P \subset A$ such that P is precompact. Let $x_0 \in P^-$. Let x_α be a net in P such that $x_\alpha \rightarrow x_0$. Now $A \cap P^-$ is a closed, precompact subset of A .

So $A \cap P^-$ is complete. Also $x_\alpha \in A \cap P^-$ for all α and x_α is a cauchy net. So there exists a $y \in A \cap P^-$ such that $x_\alpha \rightarrow y$. This implies $y = x_0$, since Z is Hausdorff. Hence $x_0 \in A$. So $P^- \subset A$. Hence A is p -closed.

2) Suppose P is a closed, precompact subset of A . Then $P = A \cap C$, where C is closed in Z . Now P^- is closed and precompact in Z , so P^- is complete. Let x_α be a cauchy net in P . Then there exists a $x_0 \in P^-$ such that $x_\alpha \rightarrow x_0$. Now $P \subset C$, so $P^- \subset C^- = C$.

Thus $x_0 \in C$. Also A is p -closed, so $P^- \subset A$. Hence $x_0 \in A$. So $x_0 \in A \cap C = P$. Recalling $x_\alpha \rightarrow x_0$, we see that P is complete. Hence A is p -complete.

QED on lemma

Now we are in a position to prove 2.6.

Let X be a Hausdorff uniform space. Let \hat{X} and $i: X \rightarrow \hat{X}$ denote the completion of X and the canonical embedding respectively. Let X^\sim equal the p -closure of $i(X)$ in \hat{X} (i.e. $i(X)^\sim$). Give X^\sim the relative uniformity from \hat{X} . Because the image of i is dense in \hat{X} and since i is an isomorphism onto its range, $i: X \rightarrow X^\sim$ has a dense image and is an isomorphism onto its range. X^\sim is p -complete by 2.6f since X^\sim is p -closed in \hat{X} and since \hat{X} is p -complete.

Now suppose Y is a p -complete uniform space and $f: X \rightarrow Y$ is uniformly continuous. Let \hat{Y} and $j: Y \rightarrow \hat{Y}$ denote the completion of Y and the canonical embedding respectively. Let $\bar{f}: \hat{X} \rightarrow \hat{Y}$ denote the unique uniformly continuous map which makes the diagram

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 i \downarrow & & \downarrow j \\
 \hat{X} & \xrightarrow{\bar{f}} & \hat{Y}
 \end{array} \quad \text{commute.}$$

By 2.6e, \bar{f} is continuous from \hat{X} to \hat{Y} when both are given their p -topologies. So

$$\bar{f}(X^\sim) = \bar{f}(i(X)^\sim) \subset \bar{f}(i(X))^\sim = j(f(X))^\sim \subset j(Y)^\sim$$

Now Y is p -complete and j is an isomorphism onto its range. So $j(Y)$ is p -complete. Hence by 2.6f, $j(Y)$ is

p -closed in Y . Hence $j(Y)^\sim = j(Y)$. So $\bar{f}(X^\sim) \subset j(Y)$. Let $k: j(Y) \rightarrow Y$ be the "inverse" of j , thus $k(j(y)) = y$ for all $y \in Y$. Let $f^\sim = k \circ (\bar{f}|_{X^\sim})$. f^\sim is uniformly continuous since $\bar{f}|_{X^\sim}$ and k are, also

$$(f^\sim \circ i)(x) = k(\bar{f}(i(x))) = k(j(f(x))) = f(x) \quad \text{for all } x \in X.$$

So $f^\sim \circ i = f$.

Now suppose $g: X^\sim \rightarrow Y$ is uniformly continuous such that $g \circ i = f$. Then $i(X) \subset \{r \in X^\sim : g(r) = f^\sim(r)\}$. So by density of $i(X)$, $f^\sim = g$. Hence f^\sim is unique.

QED

2.7 Corollary. The category of p -complete uniform spaces and uniformly continuous maps in a bireflective subcategory of the category of all Hausdorff uniform spaces and uniformly continuous maps.

QED

Following are two technical results which will be needed in the near future.

2.8 Lemma. Let X and Y be topological spaces. Let $i: X \rightarrow Y$ be a homeomorphism onto its range. Suppose $i(X)$ is dense in Y . Then if U is a closed subset of Y , we have $i^{-1}(\text{int } U) = \text{int}[i^{-1}(U)]$, where int denotes the interior operation on sets.

Proof: i is continuous, so $i^{-1}(\text{int } U)$ is open in X . Hence $\text{int}[i^{-1}(\text{int } U)] = i^{-1}(\text{int } U)$. Now $i^{-1}(\text{int } U) \subset i^{-1}(U)$, so $i^{-1}(\text{int } U) = \text{int}[i^{-1}(\text{int } U)] \subset \text{int}[i^{-1}(U)]$.

To prove $\text{int}[i^{-1}(U)] \subset i^{-1}(\text{int } U)$, notice that $i(\text{int}[i^{-1}(U)])$ is open in $i(X)$ with

the relative topology. So there exists a V open in Y such that $i(\text{int}[i^{-1}(U)]) = V \cap i(X)$.

Thus $V \cap i(X) \subset U$.

Claim: $V \subset (V \cap i(X))^-$.

Let $p \in V$. Let T be an open neighborhood of p . Then $p \in V \cap T$. Since $i(X)$ is dense in Y , $i(X) \cap (V \cap T) = (i(X) \cap V) \cap T$ is non-empty. So $p \in (i(X) \cap V)^-$. QED on claim

But

$i(\text{int}[i^{-1}(U)]) = i(X) \cap V \subset V \subset (V \cap i(X))^- \subset U^- \subset U$, and V is open. Hence $i(\text{int}[i^{-1}(U)]) \subset \text{int } U$. QED

2.9 Corollary. Let X, Y , and i be as in 2.8. Let Z be a uniform space and A be a collection of continuous functions from Y to Z . Let $x_0 \in X$. Then A is equicontinuous at $i(x_0)$ iff $\{f \circ i : f \in A\}$ is equicontinuous at x_0 .

Proof: Note that if $U \subset Z \times Z$ and $z \in Z$, $U[z]$ is equal by definition to $\{y \in Z : (z, y) \in U\}$. Also note that (\Rightarrow) is trivial.

To prove (\Leftarrow) , let U be a closed member of the uniformity on Z . Then

$x_0 \in \text{int}[\cap\{(f \circ i)^{-1}(U[f \circ i(x_0)]) : f \in A\}]$, since $A \circ i$ is equicontinuous at x_0 . Hence

$$x \in \text{int}[\cap\{i^{-1}(U[f(i(x_0))]) : f \in A\}] = \text{int}[\cap\{(f \circ i)^{-1}(U[f \circ i(x_0)]) : f \in A\}].$$

But for all $f \in A$, $f: Y \rightarrow Z$ is continuous and $U[f(i(x_0))]$ is closed. So

$\cap\{f^{-1}(U[f(i(x_0))]) : f \in A\}$ is closed. So by 2.8, $x_0 \in i^{-1}(\text{int}[\cap\{f^{-1}(U[f(i(x_0))]) : f \in A\}])$. Thus A is equicontinuous at $i(x_0)$. QED

We are now ready to specialize to locally convex spaces.

2.10 Theorem. If E is a Hausdorff locally convex topological vector space, then there exists a p -complete locally convex topological vector space E^\sim and a continuous linear map $i: E \rightarrow E^\sim$ such that if F is a p -complete locally convex topological vector space and $f: E \rightarrow F$ is a continuous linear map, then there exists a unique continuous linear map $f^\sim: E^\sim \rightarrow F$ such that

$$\begin{array}{ccc}
 E & \xrightarrow{f} & F \\
 \downarrow i & \nearrow f^\sim & \\
 E^\sim & &
 \end{array}
 \quad \text{commutes.}$$

Moreover, i is an isomorphism onto its range and the range of i is dense in E^\sim .

Proof: The proof is a corollary of the proof of theorem 2.6. In that proof we may assume that \hat{X} is a Hausdorff locally convex space and that $i: X \rightarrow \hat{X}$ is linear.

Then remembering that continuous linear maps are uniformly continuous, we will be done provided it can be demonstrated that

2.10a Lemma. If E is a topological vector space and A is a linear subspace of E , then the p -closure of A

is a linear subspace of E .

Proof of lemma: Temporarily fix $x \in A$. Define

$Q: E \rightarrow E$ by $Q(y) = x + y$. Q is uniformly continuous.

Hence by 2.6e, Q is continuous from the p -topology of E to the p -topology of E . Now $Q(A) \subset A$, since A is a subspace. So $Q(\tilde{A}) \subset Q(A) \subset A$, where \tilde{A} denotes the p -closure of A . So for all $y \in \tilde{A}$, $x + y \in A$.

Hence for all $x \in A$ and $y \in \tilde{A}$, $x + y \in A$.

Temporarily fix $y \in \tilde{A}$ and define $P: E \rightarrow E$ by $P(x) = x + y$. P is uniformly continuous, hence p -continuous. So $P(\tilde{A}) \subset P(A) \subset A$. But by the above $P(A) \subset \tilde{A}$. So $P(\tilde{A}) \subset \tilde{\tilde{A}} = \tilde{A}$. So for all $x \in \tilde{A}$, $x + y \in \tilde{A}$. So for all $x, y \in \tilde{A}$, $x + y \in \tilde{A}$.

In a similar manner one can show that for all $\alpha \in \mathbb{K}$ and $x \in \tilde{A}$, $\alpha x \in \tilde{A}$. Hence \tilde{A} , the p -closure of A , is a linear subspace of E . QED on lemma

QED

2.11 Corollary. \mathcal{B} is a bireflective subcategory of \mathcal{C} .

Proof: This summarizes the information of 2.10 except that it only says that in each case the map $i: E \rightarrow \tilde{E}$ is an injective continuous linear map with dense image, i.e. it doesn't capture the fact that i is an isomorphism onto its range. QED

2.12 Corollary. Let E be a locally convex Hausdorff space and F be a p -complete locally convex space. Let

$R: \text{hom}(E^{\sim}, F) \longrightarrow \text{hom}(E, F)$ be defined by $T \longmapsto T \circ i$. Then R is an isomorphism of vector spaces and R induces a bijection between the equicontinuous sets of $\text{hom}(E^{\sim}, F)$ and those of $\text{hom}(E, F)$.

Proof: That R is a vector space isomorphism follows immediately from theorem 2.10. R thus induces a bijection between the power set of $\text{hom}(E^{\sim}, F)$ and the power set of $\text{hom}(E, F)$. So now the question boils down to whether R sends an equicontinuous subset of $\text{hom}(E^{\sim}, F)$ to an equicontinuous subset of $\text{hom}(E, F)$ and whether R^{-1} sends an equicontinuous subset of $\text{hom}(E, F)$ to an equicontinuous subset of $\text{hom}(E^{\sim}, F)$.

The answer to the first question is trivially yes. So let us work on the second. Let A be an equicontinuous subset of $\text{hom}(E, F)$. Then for all $a \in A$, $a = R(R^{-1}(a)) = [R^{-1}(a)] \circ i$. Now applying 2.9, we find that $R^{-1}(A)$ is equicontinuous. QED

2.13 Corollary. If E is a locally convex Hausdorff space, then the E^{\sim} and $i: E \longrightarrow E^{\sim}$ of theorem 2.10 may be chosen so that

1) There exists a locally convex topology \mathcal{J} on E' such that $E^{\sim} = (E', \mathcal{J})'_e$ where the subscript e denotes the topology of uniform convergence on equicontinuous subsets of E' ; and

2) $i: E \longrightarrow E^{\sim}$ is just the evaluation map.

Proof: In the last corollary, let $F = \mathbb{K}$. Note that in

this case $R = {}^t i$. Thus we see that there is a locally convex topology \mathcal{J} on E' such that $R = {}^t i$ is a topological linear isomorphism from $(E^\sim)^P$ to (E', \mathcal{J}) . Since $R = {}^t i$ induces a bijection from the equicontinuous sets of $(E^\sim)'$ to those of E' , ${}^t R = {}^t t i$ is a topological linear isomorphism from $(E', \mathcal{J})'_e$ to $((E^\sim)^P)'_e$ where the subscript "e" denotes the topology of uniform convergence on equicontinuous sets. Since E^\sim is p-complete, the evaluation map $j: E^\sim \rightarrow ((E^\sim)^P)'_e$ is a topological linear isomorphism. Also if $k: E \rightarrow (E', \mathcal{J})'_e$ is the evaluation map, then k is continuous and

$$\begin{array}{ccc}
 E & \xrightarrow{i} & E^\sim \\
 k \downarrow & & \downarrow j \\
 (E', \mathcal{J})'_e & \xrightarrow{{}^t t i} & ((E^\sim)^P)'_e
 \end{array} \text{ commutes.}$$

Now $(E', \mathcal{J})'_e$ is p-complete since E^\sim is, and since j and ${}^t t i$ are topological linear isomorphisms.

These facts together with the above diagram commuting tell us all that we need to know. QED

2.14 Definition. Let $\alpha: \mathcal{C}_n \rightarrow \mathcal{O}_n$ denote the coreflection functor (cf. 2.2). We thus have $\text{hom}(E, F) = \text{hom}(E, \alpha(F))$ for all $E \in \text{ob} \mathcal{O}_n$ and $F \in \text{ob} \mathcal{C}_n$. α will sometimes be referred to as the p-determination functor.

Let $\beta: \mathcal{C} \rightarrow \mathcal{B}$ denote the reflection functor (cf. 2.11). We thus have $\text{hom}(\beta(E), F)$ naturally isomorphic to $\text{hom}(E, F)$ for all $E \in \text{ob} \mathcal{C}$ and $F \in \text{ob} \mathcal{B}$. β will sometimes be referred to as the p-completion functor.

2.15 Theorem. If $E \in \text{ob}\mathcal{C}$, then $\alpha(E) \in \text{ob}\mathcal{O}$. If $E \in \text{ob}\mathcal{B}$, then $\alpha(E) \in \text{ob}\mathcal{D}$. If $E \in \text{ob}\mathcal{M}$, then $\beta(E) \in \text{ob}\mathcal{D}$.

Proof: If E is Hausdorff, then by 2.1 the topology of $\alpha(E)$ is finer than that of E . Hence $\alpha(E)$ is also Hausdorff.

Suppose E is p -complete and P is a closed, precompact subset of $\alpha(E)$. Then P is precompact in E . Let P^- denote the closure of P in E . P^- is closed and precompact in E , hence P^- is complete in E . By 2.1, the uniformity induced on P^- by E agrees with the one induced by $\alpha(E)$. So P^- is complete in $\alpha(E)$. But $P \subset P^-$ and is closed. So P is complete in $\alpha(E)$. Hence $\alpha(E)$ is p -complete. Thus $\alpha(E)$ is p -reflexive.

Suppose E is p -determined and Hausdorff. Let $i: E \rightarrow \beta(E)$ denote the canonical map. Then ${}^t i: [\beta(E)]^P \rightarrow E^P$ is continuous. Suppose P is a precompact subset of $[\beta(E)]^P$, then ${}^t i(P)$ is precompact in E^P . Hence ${}^t i(P)$ is equicontinuous, since E is p -determined. But then P is equicontinuous by 2.12 because ${}^t i = R$ in that proof. Hence $\beta(E)$ is p -determined and thus p -reflexive. QED

2.16 Corollary. \mathcal{D} is a full bicoreflective subcategory of \mathcal{B} , \mathcal{D} is a full bireflective subcategory of \mathcal{M} , and \mathcal{M} is a full bicoreflective subcategory of \mathcal{C} . QED

I now prove a technical lemma which will be used several times in the sequel.

2.17 Lemma. Let E be a p -determined space and F any locally convex space. If $Q \subset \text{hom}_p(E, F)$ is precompact, then the relative uniformities induced on Q by $\text{hom}_p(E, F)$ and by $\text{hom}_p(E, \alpha(E))$ are identical; in particular Q is precompact in $\text{hom}_p(E, \alpha(F))$ and therefore equicontinuous as a set of functions from E to $\alpha(F)$.

Proof: The canonical bilinear form b from $E \times \text{hom}_p(E, F)$ to F is $\mathcal{C} - \mathcal{X}$ hypocontinuous, where \mathcal{C} is all precompact subsets of E and \mathcal{X} is all equicontinuous subsets of $\text{hom}_p(E, F)$. Let Q be precompact in $\text{hom}_p(E, F)$. Q is then equicontinuous by 1.6. Let P be precompact in E and U be a neighborhood of zero in $\alpha(F)$.

$b|_{(P \times Q)}$ is uniformly continuous (cf. prop. 5, §4, chap. 3, of [4]), so $b(P \times Q) = Q(P)$ is precompact in F . By 2.1, the relative uniformities on $Q(P)$ induced by F and by $\alpha(F)$ agree, so there exists a neighborhood V of zero in F such that

$$\{(x, y) \in Q(P) \times Q(P) : x - y \in V\} \subset \{(x, y) \in Q(P) \times Q(P) : x - y \in U\}$$

So $\{(f, g) \in Q \times Q : f(p) - g(p) \in V \text{ for all } p \in P\} \subset$

$$\{(f, g) \in Q \times Q : f(p) - g(p) \in U \text{ for all } p \in P\}.$$

So the relative uniformity induced on Q by $\text{hom}_p(E, \alpha(F))$ is coarser than that induced by $\text{hom}_p(E, F)$.

The other inclusion is trivial. Hence they agree.

The last statement in 2.17 follows from 1.6. QED

Section 3 - The category of p-reflexive spaces is
 "strongly" self-dual.

The main results of this section are corollary 3.5, theorem 3.8, and corollaries 3.9 and 3.10.

The word "strong" is used to remind the reader that the Hom functor defined below takes its values in the category of p-reflexive spaces.

3.1 Definition. If E and F are locally convex spaces, let $\text{Hom}(E,F) = \alpha(\text{hom}_p(E,F))$ where $\alpha: \mathcal{C} \rightarrow \mathcal{A}$ is the coreflection functor.

Note that Hom is a functor from $\mathcal{C}^{\text{op}} \times \mathcal{C}$ to \mathcal{A} and that by 1.23, hom_p can be considered as a functor from $\mathcal{A}^{\text{op}} \times \mathcal{B}$ to \mathcal{B} . Thus Hom can be considered as a functor from $\mathcal{A}^{\text{op}} \times \mathcal{B}$ to \mathcal{D} or from $\mathcal{D}^{\text{op}} \times \mathcal{D}$ to \mathcal{D} (cf. 2.15).

Also define functors $\text{hom}^{\text{op}}: \mathcal{C} \times \mathcal{C}^{\text{op}} \rightarrow \text{Sets}$, $\text{hom}_p^{\text{op}}: \mathcal{C} \times \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$, and $\text{Hom}^{\text{op}}: \mathcal{C} \times \mathcal{C}^{\text{op}} \rightarrow \mathcal{A}$; by $\text{hom}^{\text{op}} = \text{hom} \circ L$, $\text{hom}_p^{\text{op}} = \text{hom}_p \circ L$, and $\text{Hom}^{\text{op}} = \text{Hom} \circ L$, where $L: \mathcal{C} \times \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{C}$ is the isomorphism of categories defined by $(r,s) \mapsto (s,r)$.

Note that $\text{hom}^{\text{op}} = \text{Mor}_{\mathcal{C}^{\text{op}}}$.

3.2 Definition. Define a functor $R: \mathcal{D} \rightarrow \mathcal{D}^{\text{op}}$ such that $R(E) = E^{\text{p}}$ and such that if $f \in \text{Mor}_{\mathcal{D}}(E,F)$, then $R(f) = {}^t f$ where ${}^t f$ denotes the transpose of f.

Let $I_{\mathcal{D}}$ be the identity functor on \mathcal{D} . For each $D \in \mathcal{D}$, define $\eta_E: E \rightarrow R^{OP}(R(E)) = E^{PP}$ by $\eta_E(e)(\phi) = \phi(e)$, i.e. the evaluation map.

3.3 Theorem. R is a functor and $\eta: I_{\mathcal{D}} \rightarrow R^{OP} \circ R$ is a natural isomorphism.

Proof: Really the only difficulty offered by a theorem like this is in proving that for all $E \in \text{ob}\mathcal{D}$, η_E is an isomorphism. But by definition, a locally convex space is in \mathcal{D} iff η_E is an isomorphism. QED

3.4 Corollary. $\eta^{OP}: R \circ R^{OP} \rightarrow I_{\mathcal{D}^{OP}}$ defined by $\eta^{OP}(E) = \eta(E)$ for all $E \in \text{ob}\mathcal{D}^{OP}$ is a natural isomorphism. QED

3.5 Corollary. The categories \mathcal{D} and \mathcal{D}^{OP} are equivalent categories and R is an equivalence. QED

The following lemma will be needed in the next theorem.

3.6 Lemma. $\eta_{R(E)}^{OP} \circ_{\mathcal{D}^{OP}} R(\eta_E) = 1_{R(E)}$ for all $E \in \text{ob}\mathcal{D}$.

Proof: $\eta_{R(E)}^{OP} \circ_{\mathcal{D}^{OP}} R(\eta_E) = R(\eta_E) \circ_{\mathcal{D}} \eta_{R(E)}^{OP}$. Let $\phi \in R(E) = E^P$. Let $e \in E$. Then

$$\begin{aligned} [[R(\eta_E) \circ_{\mathcal{D}} \eta_{R(E)}^{OP}](\phi)](e) &= [R(\eta_E)(\eta_{R(E)}^{OP}(\phi))](e) \\ &= [[\eta_{R(E)}^{OP}(\phi)] \circ_{\mathcal{D}} \eta_E](e) \\ &= \eta_{R(E)}^{OP}(\phi)(\eta_E(e)) \\ &= [\eta_E(e)](\phi) \\ &= \phi(e) \end{aligned}$$

Thus $[R(\eta_E) \circ_{\mathcal{D}} \eta_{R(E)}^{OP}](\phi) = \phi$. So $\eta_{R(E)}^{OP} \circ_{\mathcal{D}^{OP}} R(\eta_E) = 1_{R(E)}$. QED

3.7 Theorem. The natural transformation induced by R from $\text{hom}(E, F)$ to $\text{hom}^{\text{OP}}(R^{\text{OP}}(E), R(F))$ induces a natural isomorphism from $\text{hom}_p(E, F)$ to $\text{hom}_p^{\text{OP}}(R^{\text{OP}}(E), R(F))$.

Proof: It follows from ordinary category theory that the canonical map \mathcal{K} from hom to $\text{hom}^{\text{OP}} \circ (R^{\text{OP}} \times R)$ is a natural transformation. So that to show that \mathcal{K} induces a natural transformation at the \mathcal{C} level, we only need show that for each $E, F \in \text{ob } \mathcal{D}$, $\mathcal{K}(E, F)$ from $\text{hom}(E, F)$ to $\text{hom}_p^{\text{OP}}(R^{\text{OP}}(E), R(F))$ is both continuous and linear. Linearity is trivially satisfied.

Claim: $\mathcal{K}(E, F)$ is continuous.

Let P be a precompact subset of $R(F) = F^{\mathbb{P}}$. Let U be a closed, convex, balanced neighborhood of zero in $R(E) = E^{\mathbb{P}}$. Let $B = \{S \in \text{hom}(R(F), R(E)) : S(P) \subset U\}$. B is a basic neighborhood of zero in $\text{hom}_p(R(F), R(E))$. Now U° is a precompact set, since there exists a precompact set $Q \subset E$ such that $Q^\circ \subset U$. Also P is equicontinuous, since F is p -determined. So P° is a neighborhood of zero in F . Let $A = \{T \in \text{hom}(E, F) : T(U^\circ) \subset P^\circ\}$. A is a neighborhood of zero in $\text{hom}_p(E, F)$.

So we are done with the claim if we can show that $\mathcal{K}(E, F)(A) \subset B$. To this end, let $T \in A$. Then $\mathcal{K}(E, F)(T) = {}^t T$. Suppose $\phi \in P$ and $x \in U^\circ$. Then ${}^t T(\phi)(x) = \phi(T(x))$. But $T(U^\circ) \subset P^\circ$, so $T(x) \in P^\circ$. Thus $|\phi(T(x))| \leq 1$, since $\phi \in P$. So ${}^t T(\phi) \in U^{\circ\circ} = U$. Thus ${}^t T(P) \subset U$. Hence $\mathcal{K}(E, F)(T) = {}^t T \in B$. Hence $\mathcal{K}(E, F)$

is continuous.

QED on claim

So \mathcal{K} is a natural transformation. Note that

$$\mathcal{K}(E,F)(f) = R(f) \quad \text{for all } f \in \text{hom}(E,F).$$

Next to show that for all $E, F \in \text{ob}\mathcal{D}$, $\mathcal{K}(E,F)$ is a \mathcal{C} -isomorphism.

For all $E, F \in \text{ob}\mathcal{D}$, define $\zeta(E,F)$ from $\text{hom}_p^{\text{OP}}(R(E), R(F))$ to $\text{hom}_p(E,F)$ by $f \mapsto \eta_F^{-1} \circ R^{\text{OP}}(f) \circ \eta_E$.

Claim: $\zeta(E,F)$ is continuous and linear.

R is linear and continuous, by the last claim.

Also η_E and η_F^{-1} are linear and continuous by 3.3.

Thus η_E sends precompact sets to precompact sets and η_F^{-1} has the property that the inverse image of neighborhoods of zero are neighborhoods of zero.

Thus by noting the definition of $\zeta(E,F)$, we see that it is both continuous and linear. QED on claim

Claim: 1) $\zeta(E,F) \circ \mathcal{K}(E,F) = 1_{\text{hom}_p(E,F)}$, and

2) $\mathcal{K}(E,F) \circ \zeta(E,F) = 1_{\text{hom}_p^{\text{OP}}(R^{\text{OP}}(E), R(F))}$.

In order to prove 1), note that if $f \in \text{hom}(E,F)$, then $[\zeta(E,F) \circ \mathcal{K}(E,F)](f) = \eta_F^{-1} \circ R^{\text{OP}}(R(f)) \circ \eta_E = f$ since $\eta: I_{\mathcal{D}} \rightarrow R^{\text{OP}} \circ R$ is a natural isomorphism.

In order to prove 2), let $f \in \text{hom}^{\text{OP}}(R^{\text{OP}}(E), R(F))$.

$$\begin{aligned} \text{Then } [\mathcal{K}(E,F) \circ \zeta(E,F)](f) &= R(\eta_F^{-1} \circ R^{\text{OP}}(f) \circ \eta_E) \\ &= R(\eta_F)^{-1} \circ_{\mathcal{D}^{\text{OP}}} R(R^{\text{OP}}(f)) \circ_{\mathcal{D}^{\text{OP}}} R(\eta_E). \end{aligned}$$

Now by lemma 3.6, we find that $R(\eta_E) = [\eta_{R(E)}^{\text{OP}}]^{-1}$ and

$R(\eta_F) = [\eta_{R(F)}^{\text{OP}}]^{-1}$. Thus

$$[\mathcal{K}(E,F) \circ \zeta(E,F)](f) = \eta_{R(F)}^{\text{OP}} \circ_{\mathcal{D}^{\text{OP}}} (R \circ R^{\text{OP}}(f)) \circ_{\mathcal{D}^{\text{OP}}} [\eta_{R(E)}^{\text{OP}}]^{-1}.$$

But $\eta^{\text{OP}}: R \circ R^{\text{OP}} \rightarrow I_{\mathcal{D}^{\text{OP}}}$ is a natural isomorphism by 3.4.

Hence by naturality we find that

$$[\mathcal{K}(E,F) \circ \zeta(E,F)](f) = f.$$

$$\text{Hence } \mathcal{K}(E,F) \circ \zeta(E,F) = 1_{\text{hom}_p^{\text{OP}}(R^{\text{OP}}(E), R(F))}.$$

QED on claim

So \mathcal{K} is a natural isomorphism from hom_p to $\text{hom}_p^{\text{OP}} \circ (R^{\text{OP}} \times R)$. QED

3.8 Theorem. For all E and F p -reflexive,

- a) $\text{hom}_p(E,F) \cong \text{hom}_p^{\text{OP}}(R^{\text{OP}}(E), R(F))$,
- b) $\text{Hom}(E,F) \cong \text{Hom}^{\text{OP}}(R^{\text{OP}}(E), R(F))$,
- c) $\text{hom}_p(R(E), F) \cong \text{hom}_p^{\text{OP}}(E, R(F))$,
- d) $\text{Hom}(R(E), F) \cong \text{Hom}^{\text{OP}}(E, R(F))$,
- e) $\text{hom}_p(E, R^{\text{OP}}(F)) \cong \text{hom}_p^{\text{OP}}(R^{\text{OP}}(E), F)$, and
- f) $\text{Hom}(E, R^{\text{OP}}(F)) \cong \text{Hom}^{\text{OP}}(R^{\text{OP}}(E), F)$;

where \cong means "is naturally isomorphic to".

Proof: b), d), and f) follow from a), c), and e) respectively by just applying the p -determination functor to everything in sight. a) is just the statement of theorem 3.7. So we need only prove c) and e).

Proof of c): Define a natural isomorphism

$$\omega: R \rightarrow R \text{ by } \omega_E = 1_{R(E)} \text{ for all } E \in \mathcal{O}. \text{ So}$$

$$\eta \times \omega: I_{\mathcal{O}} \times R \rightarrow (R^{\text{OP}} \circ R) \times R \text{ is a natural isomorphism}$$

(note: R and η are as in 3.3). So by A1,

$$\text{hom}_p^{\text{OP}} \circ (\eta \times \omega) : \text{hom}_p^{\text{OP}} \circ (I_{\mathcal{O}} \times R) \rightarrow \text{hom}_p^{\text{OP}} \circ ((R^{\text{OP}} \circ R) \times R)$$

is a natural isomorphism. Let $\zeta: \text{hom}_p^{\text{OP}} \circ (R^{\text{OP}} \times R) \rightarrow \text{hom}_p$

be the natural isomorphism from a). Then

$$\zeta \circ (R \times I_{\mathcal{O}}) : \text{hom}_p^{\text{OP}} \circ ((R^{\text{OP}} \circ R) \times R) \rightarrow \text{hom}_p \circ (R \times I_{\mathcal{O}}) \text{ is}$$

a natural isomorphism again by A1.

So the composition of these things gives a natural isomorphism from $\text{hom}_P^{\text{OP}} \circ (I_P \times R)$ to $\text{hom}_P(R \times I_Q)$ (cf. A2).

Proof of e): It is entirely analogous to that of c). QED

3.9 Corollary. R^{OP} is both an adjoint and a coadjoint of R .

3.10 Corollary. R preserves both limits and colimits.

Proof of 3.9 and 3.10: Following statement c) of 3.8 by the forgetful functor, we see that R^{OP} is an adjoint of R . By following statement e) of 3.8 by the forgetful functor, we find that R^{OP} is a coadjoint of R . R having an adjoint implies that R preserves limits; and R having a coadjoint implies that R preserves colimits.

QED

Section 4 - Limits and colimits in the category of p-reflexive spaces.

Theorem 4.9 proves that limits and colimits in \mathcal{D} always exist (i.e. that the category is both complete and cocomplete) and shows how to compute them in terms of the category of locally convex Hausdorff spaces. In particular, corollary 4.4 shows that products and coproducts in \mathcal{D} agree with the products and coproducts in \mathcal{C} .

Recall that I have already proved that \mathcal{B} is complete, \mathcal{O} is cocomplete, and that the limits in \mathcal{B} agree with the limits in \mathcal{C} and the colimits in \mathcal{O} agree with the colimits in \mathcal{C} .

4.1 Theorem. Suppose I is an index set and for each $\alpha \in I$, B_α is a p-complete space. Then $\bigoplus_{\alpha \in I} B_\alpha$, the locally convex direct sum of the B_α , is p-complete.

Proof: Suppose P is a closed precompact subset of $\bigoplus_{\alpha \in I} B_\alpha$. By (4) of §18.5 of [27], there exists a finite subset $J \subset I$ such that $P \subset \bigoplus_{\alpha \in J} B_\alpha$. Now by (2) of §18.5 of [27], the relative topology on $\bigoplus_{\alpha \in J} B_\alpha$ is just the direct sum topology. So P is closed and precompact in $\bigoplus_{\alpha \in J} B_\alpha = \prod_{\alpha \in J} B_\alpha$ since J is finite (cf. Prop. 7, §4, Chap. 2 of [3]). But $\prod_{\alpha \in J} B_\alpha$ is p-complete. So P is

complete in $\prod_{\alpha \in J} B_\alpha = \bigoplus_{\alpha \in J} B_\alpha$. So P is complete in $\bigoplus_{\alpha \in I} B_\alpha$.
 QED

4.2 Theorem. Suppose I is an index set and for each $\alpha \in I$, A_α is a p -determined Hausdorff space. Then $\prod_{\alpha \in I} A_\alpha$ is p -determined.

Proof: First we will need a lemma.

4.2a Lemma. Let I be an index set. For each $\alpha \in I$, let E_α be a locally convex Hausdorff space. Let B be a barrel in $\prod_{\alpha \in I} E_\alpha$. Then there exists a finite subset

J of I such that if $t: \prod_{\alpha \in I} E_\alpha \rightarrow \prod_{\alpha \in I} E_\alpha$ is the continuous linear projection defined by

$$t(f)(\alpha) = \begin{cases} f(\alpha) & \text{if } \alpha \in J \\ 0 & \text{if } \alpha \notin J \end{cases},$$

then we have $t^{-1}(B) = B$.

Proof of lemma: Let $E = \prod_{\alpha \in I} E_\alpha$. Then $E' = \bigoplus_{\alpha \in I} E_\alpha'$ as sets, with the duality expressed by $\langle \phi, e \rangle = \sum \phi_\alpha(e(\alpha))$

if $\phi \in \bigoplus_{\alpha \in I} E_\alpha'$ and $e \in \prod_{\alpha \in I} E_\alpha$ (cf. (2) of §22.5 of [27]).

Let B be a barrel in $\prod_{\alpha \in I} E_\alpha$. Then $B^\circ \subset E'$ is $\sigma(E', E)$

bounded. Let $R_\alpha = (E_\alpha', \sigma(E_\alpha', E_\alpha))$. Let $R = \bigoplus_{\alpha \in I} R_\alpha$.

Then $R' = \prod_{\alpha \in I} R_\alpha'$ as sets, and of course $R_\alpha' = E_\alpha$. So

$R' = E$ by (2) of §22.5 of [27]. Also $(E', \sigma(E', E))'$

equals E . Note that the topology of R is finer than

$\sigma(E', E)$, since each projection from $(E_\alpha', \sigma(E_\alpha', E_\alpha))$ to

$(E', \sigma(E', E))$ is continuous. By Mackey's theorem (thm.

3, §3.5 of [19]), $\sigma(E', E)$ and R have the same bounded

sets, so B° is bounded in R . Hence by §18.5, (4), [27],

there exists a finite subset J of I such that

$B^\circ \subset \bigoplus_{\alpha \in J} R_\alpha$.

Claim: If $\phi \in B^\circ \cap E'$, then $\phi \circ t = \phi$.

Let $f \in \prod_{\alpha \in I} E_\alpha$. Then for all $j \in J$, there exists a $\phi_j \in R_j$ such that $\phi = \Sigma\{\phi_j : j \in J\}$. Now

$$\phi(f) = \sum_{j \in J} \phi_j(f(j)) = \sum_{j \in J} \phi_j(t(f)(j)) = \phi(t(f)).$$

QED on claim

Now suppose $b \in E$. Let $\phi \in B^\circ$. Then
 $|\phi(t(b))| = |\phi(b)|$. Thus $t^{-1}(B) \subset B^{\circ\circ}$, and $B \subset t^{-1}(B^{\circ\circ})$.
 But B is a barrel, so $B^{\circ\circ} = B$. Hence $t^{-1}(B) = B$.

QED on lemma

Now returning to the proof of 4.2, suppose

$B \subset \prod_{\alpha \in I} A_\alpha$ is closed, convex, and balanced, and that
 $B \cap P$ is a neighborhood of zero in the relative topology
 on P , for all P precompact which contains zero. Then
 B is point absorbing by 1.2 and 1.3. Hence B is a
 barrel. Hence by 4.2a, there exists a finite set
 $J \subset I$ such that if $r: \prod_{\alpha \in I} A_\alpha \rightarrow \prod_{\alpha \in J} A_\alpha \cong \bigoplus_{\alpha \in J} A_\alpha$ and
 $s: \bigoplus_{\alpha \in J} A_\alpha \rightarrow \prod_{\alpha \in I} A_\alpha$ are the canonical continuous linear maps,
 then $(s \circ r)^{-1}(B) = B$. Now $s^{-1}(B)$ is closed, convex,
 and balanced. Let Q be a precompact set in $\bigoplus_{\alpha \in J} A_\alpha$
 containing zero. Then $s(Q)$ is precompact in $\prod_{\alpha \in I} A_\alpha$
 containing zero. So there exists a neighborhood V of
 zero such that $s(Q) \cap V \subset B$. $s^{-1}(V)$ is a neighborhood
 of zero in $\bigoplus_{\alpha \in J} A_\alpha$ and $s^{-1}(V) \cap Q \subset s^{-1}(B) \cap Q$. Since
 the direct sum of p -determined spaces is again p -deter-
 mined, $s^{-1}(B)$ is a neighborhood of zero. Hence
 $r^{-1}(s^{-1}(B)) = (s \circ r)^{-1}(B)$ is a neighborhood of zero. But
 $(s \circ r)^{-1}(B) = B$. So B is a neighborhood of zero. Hence
 $\prod\{A_\alpha : \alpha \in I\}$ is p -determined. QED

4.3 Corollary. The locally convex product and direct sum of p -reflexive spaces are p -reflexive.

Proof: 1.11 and 1.15 tell us that the product of p -complete spaces is p -complete and the sum of p -determined spaces is p -determined. So the result follows from 4.1, 4.2, and 1.21. QED

4.4 Corollary. Products and coproducts exist in the categories \mathcal{O} , \mathcal{B} , and \mathcal{D} ; and they agree with the products and coproducts respectively in the category \mathcal{C} . QED

We will now investigate kernels in the category \mathcal{O} and cokernels in the category \mathcal{B} .

4.5 Theorem. If R and S are in $\text{ob } \mathcal{B}$ and $f: R \rightarrow S$ is a morphism, then the following two statements about f are equivalent:

- 1) f is the \mathcal{B} -cokernel of some morphism.
- 2) f induces an isomorphism from $(R/\ker f)^\sim$ to S , where $R/\ker f$ denotes the quotient in \mathcal{C} and $^\sim$ denotes the p -completion.

4.6 Theorem. If R and S are in $\text{ob } \mathcal{O}$ and $f: R \rightarrow S$ is a morphism, then the following two statements are equivalent:

- 1) f is the \mathcal{O} -kernel of some morphism.
- 2) $f(R)$ is closed in S and f induces an isomorphism from R to $[f(R)]^\wedge$, where $f(R)$ has the

relative topology from S and $\hat{\cdot}$ denotes the p -determination.

Proof of 4.5: (1 \Rightarrow 2) Suppose f is a cokernel of a map $p: Q \rightarrow R$. Then $f \circ p = 0$. So $p(Q) \subset \ker f$. Let $r: R \rightarrow R/\ker f$ and $s: R/\ker f \rightarrow (R/\ker f)^\sim$ be the canonical maps. So we have $(s \circ r) \circ p = 0$. Let $\bar{f}: R/\ker f \rightarrow S$ be the canonical map such that

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ r \downarrow & \nearrow \bar{f} & \\ R/\ker f & & \end{array} \quad \text{commutes.}$$

Now S is p -complete, so let $\bar{\bar{f}}: (R/\ker f)^\sim \rightarrow S$ be the unique map such that

$$\begin{array}{ccc} R/\ker f & \xrightarrow{\bar{f}} & S \\ s \downarrow & \nearrow \bar{\bar{f}} & \\ (R/\ker f)^\sim & & \end{array} \quad \text{commutes.}$$

Thus $\bar{\bar{f}} \circ (s \circ r) = f$. Note that $s \circ r$ is an epimorphism since both s and r are. Since $(s \circ r) \circ p = 0$ and since f is a cokernel of p , there exists a unique map $g: S \rightarrow (R/\ker f)^\sim$ such that

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ s \circ r \downarrow & \nearrow g & \\ (R/\ker f)^\sim & & \end{array} \quad \text{commutes}$$

Hence $(g \circ \bar{\bar{f}}) \circ (s \circ r) = s \circ r = 1_{(R/\ker f)^\sim} \circ (s \circ r)$ and $(\bar{\bar{f}} \circ g) \circ f = f = 1_S \circ f$. Thus $\bar{\bar{f}} \circ g = 1_S$ and $g \circ \bar{\bar{f}} = 1_{(R/\ker f)^\sim}$, since f and $s \circ r$ are epic. So

\bar{f} is an isomorphism.

QED on $1 \Rightarrow 2$

(2 \Rightarrow 1) Suppose the \bar{f} described in the preceding is an isomorphism. Let $Q = \ker f$. Q is closed in R with the relative topology. Hence Q is p -complete. Let $p: Q \rightarrow R$ be the inclusion map. I claim that f is the cokernel of p .

Obviously $f \circ p = 0$. Suppose $\alpha: R \rightarrow Y$ is any map such that $\alpha \circ p = 0$. Then $\ker f = p(Q) \subset \ker \alpha$. So there exists $\bar{\alpha}: (R/\ker f) \rightarrow Y$ such that

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & Y \\ r \downarrow & \nearrow \bar{\alpha} & \\ R/\ker f & & \end{array} \quad \text{commutes,}$$

where r is as in the last part of the proof. Now since Y is p -complete, there exists a unique $\bar{\bar{\alpha}}: (R/\ker f)^\sim \rightarrow Y$ such that

$$\begin{array}{ccc} R/\ker f & \xrightarrow{\bar{\alpha}} & Y \\ s \downarrow & \nearrow \bar{\bar{\alpha}} & \\ (R/\ker f)^\sim & & \end{array} \quad \text{commutes,}$$

where s is as in the last part of the proof. Thus $\bar{\bar{\alpha}} \circ (s \circ r) = \alpha$. Let $\beta = \bar{\bar{\alpha}} \circ \bar{f}^{-1}$. Now $\beta \circ f = \beta \circ \bar{f} \circ (s \circ r) = \bar{\bar{\alpha}} \circ (s \circ r) = \alpha$.

Now suppose $\beta': S \rightarrow Y$ and $\beta' \circ f = \alpha$. Then $\beta' \circ f = \beta \circ f$, so $\beta' = \beta$ since f is epic (f is epic because $f = \bar{f} \circ (s \circ r)$ and both \bar{f} and $s \circ r$ are epic). So f is a cokernel of p .

QED on 4.5

Proof of 4.6: (1 \Rightarrow 2) Suppose f is the kernel of $p: S \rightarrow T$. Give $f(R)$ the relative topology from S . Let $Y =$ the closure of $f(R)$ in S with the relative topology. Let $i: R \rightarrow Y$ be defined by $i(r) = f(r)$. Let $j: Y \rightarrow S$ be the inclusion map. Let Y^\wedge denote the p -determination of Y and $k: Y^\wedge \rightarrow Y$ be the canonical map. Let $\ell: R \rightarrow Y^\wedge$ be the unique map such that

$$\begin{array}{ccc} R & \xrightarrow{i} & Y \\ & \searrow \ell & \uparrow k \\ & & Y^\wedge \end{array} \quad \text{commutes.}$$

We already know that $j \circ i = f$, thus $(j \circ k) \circ \ell = f$.

Now $p \circ f = 0$ because f is a kernel of p . So $f(R) \subset \ker p$. So $Y = f(R)^- \subset \ker p$. So if $x \in Y^\wedge$, then $j(k(x)) \in \ker p$. Thus $p \circ (j \circ k)$ equals 0.

Hence because f is a kernel of p , there exists a $g: Y^\wedge \rightarrow R$ such that

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ & \searrow g & \uparrow j \circ k \\ & & Y^\wedge \end{array} \quad \text{commutes.}$$

So $(j \circ k) \circ (\ell \circ g) = f \circ g = j \circ k = (j \circ k) \circ 1_{Y^\wedge}$ and $f \circ (g \circ \ell) = f = f \circ 1_R$. But f is monic because it is a kernel, and $j \circ k$ is monic because both j and k are. So $g \circ \ell = 1_R$ and $\ell \circ g = 1_{Y^\wedge}$. So ℓ is an isomorphism.

Hence I will be done if I can show that $f(R)$ is

closed in S . To this end, let $x \in f(R)^{\bar{}} = Y$. k is surjective, so there exists a $y \in Y^{\hat{}}$ such that $k(y) = x$. Now $\ell(g(y)) = y$. So $(k \circ \ell)(g(y)) = k(y)$. Hence $i(g(y)) = x$. But $i(g(y)) \in f(R)$. So $x \in f(R)$. Thus $f(R)^{\bar{}} \subset f(R)$. QED on $1 \Rightarrow 2$

In order to show $2 \Rightarrow 1$, suppose $f(R)$ is closed in S and that $\ell: R \rightarrow Y^{\hat{}}$ described in the last part is an isomorphism. Since $f(R)$ is closed, $Y = f(R)$ with the relative topology from S . Let $T = S/f(R)$. Let $p: S \rightarrow T$ be the canonical projection. I claim f is the kernel of p . We have that $p \circ f = 0$. Suppose $\gamma: Z \rightarrow S$ is such that $p \circ \gamma = 0$. Then $\gamma(Z) \subset \ker p = f(R) = Y$. Let $j: Y \rightarrow S$ be the map described in the proof of $1 \Rightarrow 2$. Define $\delta: Z \rightarrow Y$ by $\delta(z) = \gamma(z)$. Then $\gamma = j \circ \delta$. So there exists a unique map $\eta: Z \rightarrow Y^{\hat{}}$ such that

$$\begin{array}{ccc} Z & \xrightarrow{\eta} & Y^{\hat{}} \\ & \searrow \delta & \downarrow k \\ & & Y \end{array} \quad \text{commutes,}$$

where k is the canonical map. Thus $(j \circ k) \circ \eta = \gamma$.

Let $\tau = \ell^{-1} \circ \eta$.

Claim: τ is the unique map such that

$$\begin{array}{ccc} Z & \xrightarrow{\gamma} & S \\ & \searrow \tau & \uparrow f \\ & & R \end{array} \quad \text{commutes.}$$

$$1) \quad f \circ \tau = ((j \circ k) \circ \ell) \circ (\ell^{-1} \circ \eta) = (j \circ k) \circ \eta = \gamma.$$

$$2) \quad \text{Suppose } \tau': Z \rightarrow R \text{ is such that } f \circ \tau' = \gamma.$$

Then using 1) we find that $\tau = \tau'$, since f is monic (f is monic because both $j \circ k$ and ℓ are monic and since $f = (j \circ k) \circ \ell$).

Thus f is the kernel of p . QED on 4.6

4.7 Theorem. In the statements of theorems 4.5 and 4.6, if \mathcal{O} and \mathcal{B} are replaced by the category \mathcal{D} throughout, then the theorems remain true.

Proof: (kernels \Rightarrow) Suppose $R, S \in \text{ob } \mathcal{D}$ and $f: R \rightarrow S$ is the \mathcal{D} -kernel of some map. Since \mathcal{D} is a full reflexive subcategory of \mathcal{O} (cf. 2.16), the forgetful functor preserves limits. Thus $f: R \rightarrow S$ is the \mathcal{O} -kernel of some map. So f induces an \mathcal{O} -isomorphism from R to $[f(R)]^\wedge$ and $f(R)$ is closed in S (by 4.6). Since $f(R)$ is closed in S , $f(R)$ is p -complete, so $f(R)^\wedge$ is p -reflexive. Since \mathcal{D} is a full subcategory, f induces a \mathcal{D} -isomorphism from R to $f(R)^\wedge$.

(kernels \Leftarrow) Suppose $f(R)$ is closed in S and f induces an \mathcal{D} -isomorphism from R to $f(R)^\wedge$. So by 4.6, f is a \mathcal{O} -kernel of some \mathcal{O} -map $p: S \rightarrow X$. $X \in \text{ob } \mathcal{O}$ and recall R and S are in $\text{ob } \mathcal{D}$. Let X^\sim denote the p -completion of X and let $t: X \rightarrow X^\sim$ denote the canonical morphism. Note $X^\sim \in \text{ob } \mathcal{D}$.

Claim: f is the \mathcal{D} -kernel of the map $t \circ p$.

$$1) \quad (t \circ p) \circ f = t \circ (p \circ f) = t \circ 0 = 0$$

2) Suppose $g: Y \rightarrow S$ such that $Y \in \text{ob}\mathcal{D}$ and $(t \circ p) \circ g = 0$, then $p \circ g = 0$ since t is monic. So there exists a unique \mathcal{M} -map $\theta: Y \rightarrow R$ such that $f \circ \theta = g$, since f is the \mathcal{M} -kernel of p . Since \mathcal{D} is full in \mathcal{M} , θ is a \mathcal{D} -map. And since every \mathcal{D} -map is an \mathcal{M} -map, f is a \mathcal{D} -kernel of $t \circ p$.

(cokernels \Rightarrow) Suppose R and S are in $\text{ob}\mathcal{D}$ and $f: R \rightarrow S$ is the \mathcal{D} -cokernel of some map. Since \mathcal{D} is a full coreflective subcategory of \mathcal{B} (cf. 2.16), the forgetful functor preserves colimits. Hence $f: R \rightarrow S$ is the \mathcal{B} cokernel of some map. So f induces a \mathcal{B} -isomorphism from $(R/\ker f)^\sim$ to S (by 4.5). $R/\ker f$ is p -determined because R is, so $(R/\ker f)^\sim$ is p -reflexive. Since \mathcal{D} is a full subcategory of \mathcal{B} , f induces a \mathcal{D} isomorphism from $(R/\ker f)^\sim$ to S .

(cokernels \Leftarrow) Suppose f induces a \mathcal{D} isomorphism from $(R/\ker f)^\sim$ to S . Then by 4.5, f is the \mathcal{B} -cokernel of some \mathcal{B} -map $p: X \rightarrow R$, where $X \in \text{ob}\mathcal{B}$. Let X^\wedge denote the p -determination of X and $t: X^\wedge \rightarrow X$ the canonical map. Note that X^\wedge is p -reflexive, since $X \in \text{ob}\mathcal{B}$. Recall R and S are in $\text{ob}\mathcal{D}$.

Claim: f is the \mathcal{D} -cokernel of $p \circ t$.

$$1) f \circ (p \circ t) = (f \circ p) \circ t = 0 \circ t = 0$$

2) Suppose $g: R \rightarrow Y$ is such that $Y \in \text{ob}\mathcal{D}$ and $g \circ (p \circ t) = 0$. Then $g \circ p = 0$, since t is epic. So there exists a unique \mathcal{B} -map $\theta: S \rightarrow Y$ such that $\theta \circ f = g$, since f is the \mathcal{B} -cokernel of p . Since \mathcal{D} is full in \mathcal{B} , θ is a \mathcal{D} -map. And since every \mathcal{D} -map is a

\mathcal{Q} -map, f is a \mathcal{D} -cokernel of $p \circ t$. QED

4.8 Corollary. Let E be a locally convex Hausdorff space and A be a closed subspace of E . Let $\hat{}$ denote p -determination and \sim denote p -completion. Let

$$A \xrightarrow{r} A, A \xrightarrow{s} E, E \xrightarrow{t} E/A, E/A \xrightarrow{u} (E/A) \sim$$

denote the canonical maps. Then

- 1) $s = \ker_{\mathcal{C}}(t)$ and $t = \text{coker}_{\mathcal{C}}(s)$;
- 2) if $E \in \mathcal{O}$, $s \circ r = \ker_{\mathcal{O}}(t)$ and $t = \text{coker}_{\mathcal{O}}(s \circ r)$;
- 3) if $E \in \mathcal{G}$, $s = \ker_{\mathcal{G}}(u \circ t)$ and $u \circ t = \text{coker}_{\mathcal{G}}(s)$;
- 4) if $E \in \mathcal{D}$, $s \circ r = \ker_{\mathcal{D}}(u \circ t)$ and
 $u \circ t = \text{coker}_{\mathcal{D}}(s \circ r)$.

Proof: In each case call the kernel statement (a) and the cokernel statement (b).

1(a) and 1(b) are folklore. 3(b) follows from the proof of $2 \Rightarrow 1$ in 4.5. 2(a) follows from the proof of $2 \Rightarrow 1$ in 4.6. 4(a) follows from the proof of (kernel \Leftarrow) in 4.7 together with 2(a). 4(b) follows from the proof of (cokernel \Leftarrow) in 4.7 together with 3(b). 2(b) follows from 1(b).^{*} 3(a) follows from 1(a) in the same manner as 4(a) follows from 2(a). QED

4.9 Theorem. $\mathcal{O}, \mathcal{G}, \mathcal{C}$, and \mathcal{D} are all complete and co-complete categories.

The limits in \mathcal{G} and \mathcal{C} agree, the colimits in \mathcal{O} and \mathcal{C} agree, the limits in \mathcal{O} and \mathcal{D} are the p -determination of limits in \mathcal{C} , and the colimits in \mathcal{G} and \mathcal{D}

^{*} - in the same manner as 4(b) follows from 3(b).

are the p-completion of the colimits in \mathcal{C} .

Proof: \mathcal{M} is a full coreflective subcategory of \mathcal{C} and \mathcal{C} is complete, hence \mathcal{M} is complete and the limits are the p-determination of the limits in \mathcal{C} (cf. A7). We know the colimits of \mathcal{C} and \mathcal{M} agree by 1.11. By the dual theorems, \mathcal{B} is cocomplete since \mathcal{C} is cocomplete and \mathcal{B} is a full reflective subcategory of \mathcal{C} . And the colimits in \mathcal{B} are the p-completion of the colimits in \mathcal{C} . We know the limits of \mathcal{B} and \mathcal{C} agree by 1.17.

So now we have proved everything except that \mathcal{D} is complete and cocomplete, and except for the description of the limits and colimits in \mathcal{D} .

Again by A7, 1.17, and 2.16, \mathcal{D} is complete because it is a full coreflective subcategory of \mathcal{B} . We also get that the limits in \mathcal{D} are the coreflections in \mathcal{D} of the limits in \mathcal{B} . But the limits in \mathcal{B} are the limits in \mathcal{C} and the coreflections in \mathcal{D} are the coreflections in \mathcal{M} . This takes care of the limits in \mathcal{D} .

Dually, because \mathcal{D} is a full reflective subcategory of the cocomplete category \mathcal{M} , \mathcal{D} is cocomplete and the colimits of \mathcal{D} are the reflection in \mathcal{D} of the colimits in \mathcal{M} . But the colimits of \mathcal{M} are the same as those in \mathcal{C} and the reflections in \mathcal{D} of elements of \mathcal{M} are the reflections in \mathcal{B} of those elements. QED

Section 5 - The duals of various limits and colimits

We have shown what various common limits and colimits in \mathcal{A} are, and we have shown how the categories \mathcal{A} and \mathcal{A}^{op} are related. In the one theorem of this section, we combine these results to show that

- 1) $(\bigoplus E_{\alpha})^{\text{p}} \cong \prod E_{\alpha}^{\text{p}}$,
- 2) $(\prod E_{\alpha})^{\text{p}} \cong \bigoplus E_{\alpha}^{\text{p}}$, and
- 3) to within a first approximation $(E/A)^{\text{p}} \cong A^{\circ}$ and $E^{\text{p}}/A^{\circ} \cong A^{\text{p}}$.

5.1 Theorem.1) Let I be an index set. For each $\alpha \in I$ let E_{α} be a p -reflexive space. Then

$$\left(\bigoplus_{\alpha \in I} E_{\alpha}\right)^{\text{p}} \cong \prod_{\alpha \in I} E_{\alpha}^{\text{p}} \quad \text{and} \quad \left(\prod_{\alpha \in I} E_{\alpha}\right)^{\text{p}} \cong \bigoplus_{\alpha \in I} E_{\alpha}^{\text{p}}.$$

2) If E is p -reflexive and A is a closed subspace of E , then

$$[(E/A)^{\sim}]^{\text{p}} \cong (A^{\circ})^{\wedge} \quad \text{and} \quad (A^{\wedge})^{\text{p}} \cong (E^{\text{p}}/A^{\circ})^{\sim},$$

where A and A° have the relative topologies induced by E and E^{p} respectively, and where \wedge denotes p -determination and \sim denotes p -completion.

Proof: We have already noted that the functor $R: \mathcal{A} \rightarrow \mathcal{A}^{\text{op}}$ defined in 3.2 preserves both limits and colimits (cf. 3.10).

In 4.4, we note that products and coproducts of \mathcal{D} agree with those of \mathcal{C} . But coproducts in \mathcal{D}^{op} are products in \mathcal{D} and products in \mathcal{D}^{op} are coproducts in \mathcal{D} . Thus 1) above follows trivially.

For 2), suppose that A is a closed subspace of E . Let $i: A \rightarrow E$ be the inclusion map. i is a \mathcal{D} -kernel by 4.8. So $\tau_i: E^{\mathcal{P}} \rightarrow (A^{\wedge})^{\mathcal{P}}$ is a cokernel. Hence τ_i induces an isomorphism from $(E^{\mathcal{P}}/\ker \tau_i)^{\sim}$ to $(A^{\wedge})^{\mathcal{P}}$ by 4.7. But $\phi \in \ker \tau_i$ iff $\phi \circ i = 0$ iff $\phi(a) = 0$ for all $a \in A$ iff $\phi \in A^{\circ}$. So $\ker \tau_i = A^{\circ}$. So this gives us one half of 2).

As for the other part of 2), consider the canonical map j from E to $(E/A)^{\sim}$. j is a \mathcal{D} -cokernel by 4.8. So $\tau_j: [(E/A)^{\sim}]^{\mathcal{P}} \rightarrow E^{\mathcal{P}}$ is a kernel in \mathcal{D} . So τ_j induces an isomorphism from $[(E/A)^{\sim}]^{\mathcal{P}}$ to $[\tau_j([(E/A)^{\sim}]^{\mathcal{P}})]^{\wedge}$ by 4.7. So I'll be done if I can prove the following

Claim: $\tau_j([(E/A)^{\sim}]^{\mathcal{P}}) = A^{\circ}$.

Suppose $\phi \in \text{hom}((E/A)^{\sim}, K)$, then for all $a \in A$, $(\phi \circ j)(a) = 0$. So $\phi \circ j \in A^{\circ}$. So $\tau_j([(E/A)^{\sim}]^{\mathcal{P}}) \subset A^{\circ}$. Also if $\phi \in A^{\circ}$, then $\phi(a) = 0$ for all $a \in A$, since A is a cone. So there exists a $\bar{\phi}: (E/A)^{\sim} \rightarrow K$ such that $\bar{\phi} \circ j = \phi$, since j is a cokernel of the canonical map from A^{\wedge} to E by 4.8. So $\tau_j(\bar{\phi}) = \phi$. Hence $A^{\circ} \subset \tau_j([(E/A)^{\sim}]^{\mathcal{P}})$.

QED on claim

QED

Section 6 - dF spaces

I will now introduce a class of spaces which are very closely related to Frechet spaces and which share many of the same properties. They are called dF spaces.

6.1 Definition. A topological space X is called hemi-compact iff there exists a countable collection \mathcal{K} of compact sets such that if $L \subset X$ is compact, then there exists a $K \in \mathcal{K}$ such that $L \subset K$.

6.2 Definition. A locally convex topological vector space E is said to be a dF space iff E is p-reflexive and hemi-compact.

Note: These dF spaces are not to be confused with the DF spaces of Grothendieck.

6.3 Theorem. 1) A locally convex space E is a Frechet space iff E is p-reflexive and has a countable base for the neighborhood system at zero.

2) A locally convex space E is a dF space iff there exists a Frechet space H such that $E \cong H^{\mathbb{P}}$.

6.4 Theorem. If E is either a Frechet or a dF space, then E is a k-space and satisfies the Krein-Smulyan property. Thus E is hypercomplete, Ptak, and complete.

(See problems 13F and 18G of [22] for a discussion of hypercompleteness).

Proof of 6.3: 1) If E is Frechet, then E is barrelled hence p -determined and there exists a countable base at zero. So suppose E is p -reflexive and there exists a countable base for the neighborhood system at zero. It then follows that E is metrizable. Also since cauchy sequences are totally bounded, p -completeness implies that E is complete.

2) (\Leftarrow) If H is a Frechet space, then H is p -reflexive and hence H^p is p -reflexive. Let \mathcal{U} be a countable base at zero for H . Let $\mathcal{K} = \{U^\circ : U \in \mathcal{U}\}$. Each element of \mathcal{K} is equicontinuous and $\sigma(H', H)$ -compact. Hence each element of \mathcal{K} is compact in H^p (cf. p. 235, [19]). Suppose C is compact in H^p . Then C is equicontinuous, since H is p -determined. So C° is a neighborhood of zero in H . So there exists a $U \in \mathcal{U}$ such that $U \subset C^\circ$. Hence $U^\circ \supset C^{\circ\circ} \supset C$. So H^p is hemi-compact. Hence H^p is a dF space.

(\Rightarrow) Suppose E is a dF space. The fact that E is hemi-compact implies that $H = E^p$ has a countable base at zero. E is p -reflexive, so H is also p -reflexive. Hence by 1) above, H is a Frechet space. Thus $H^p = E^{pp}$ is isomorphic to E , since E is p -reflexive. QED on 6.3

Proof of 6.4: It is well-known that Frechet spaces satisfy the Krein-Smulyan property and are k -spaces.

Suppose E is a dF space. Thus by 6.3, without loss of generality, we may assume that $E = H^{\mathbb{P}}$ where H is a Frechet space. First I'll show that E is a k -space.

Suppose $A \subset E$ is a set such that $A \cap K$ is closed in E for all $K \subset E$ which are compact. Let $M \subset H'$ be $\sigma(H', H)$ -closed, balanced, convex, and equicontinuous. Now equicontinuous sets are $\sigma(H', H)$ relatively compact. So M is $\sigma(H', H)$ -compact. But for equicontinuous sets, $\sigma(H', H)$ and the topology of precompact convergence agree. So M is compact in $H^{\mathbb{P}} = E$. So $A \cap M$ is closed, by hypothesis. Hence $A \cap M$ is compact since M is compact. So $A \cap M$ is $\sigma(H', H)$ -compact, hence $A \cap M$ is $\sigma(H', H)$ -closed. So by the Banach-Dieudonne theorem, A is closed in $H^{\mathbb{P}}$ (cf. cor. on p. 245 of [19]). But $E = H^{\mathbb{P}}$. Hence E is a k -space.

Now I'll show that the dF space E satisfies the Krein-Smulyan property; i.e. if $A \subset E'$ is convex and for every balanced, convex, $\sigma(E', E)$ -closed, equicontinuous subset M of E' , the set $A \cap M$ is $\sigma(E', E)$ -closed; then A is $\sigma(E', E)$ -closed.

$E \cong H^{\mathbb{P}}$ where H is a Frechet space. So $E^{\mathbb{P}} \cong H^{\mathbb{P}\mathbb{P}} \cong H$. Thus without loss of generality we may assume that $E^{\mathbb{P}} = H$. Let K be compact in H . K° is a neighborhood of zero in E , so $K^{\circ\circ}$ is equicontinuous, balanced, convex, and $\sigma(E', E)$ -closed. So $K^{\circ\circ} \cap A$ is $\sigma(E', E)$ -closed, i.e. $\sigma(H, H')$ -closed. So it is closed in H . So $K \cap (K^{\circ\circ} \cap A) = K \cap A$ is closed in H . But H is

a k -space. Hence A is closed in H . But closed, convex sets are the same for all locally convex topologies with the same dual, hence A is closed for $\sigma(H, H') = \sigma(E', E)$. Thus E has the Krein-Smulyan property.

See page 247 of [19] for a discussion of why Krein-Smulyan \Rightarrow hypercomplete \Rightarrow Ptak \Rightarrow complete. QED

The Baire category theorem implies that, in general, dF spaces are very unmetrizable.

6.5 Proposition. If E is a metrizable dF space, then E is finite dimensional.

Proof: E is complete since it is a dF space, thus E is a Frechet space. E is also hemi-compact, hence σ -compact. By the Baire category theorem we see that E has a compact neighborhood. Thus E is locally compact. Hence E must be finite dimensional. QED

6.6 Definition. A locally convex space E is said to be a dB space iff E is p -reflexive and there exists a compact subset of E which absorbs all compact subsets of E .

6.7 Theorem. 1) A locally convex space E is a Banach space iff E is p -reflexive and E has a bounded neighborhood of zero.

2) A locally convex space E is a dB space iff there exists a Banach space H such that $H^p \cong E$.

3) If E is a dB space, then E is a dF space.

Pf: 1) If E is a Banach space, then E is Frechet. So E is p -reflexive by 6.3. Also Banach spaces have bounded neighborhoods of zero.

Suppose E is a p -reflexive space which has a bounded neighborhood of zero. Then E is normable, hence metrizable. Thus E has a countable base at zero. So by 6.3, E is a Frechet space, hence complete. Thus E is a Banach space. Hence 1) is proved.

2) (\Leftarrow) Let T be a bounded neighborhood of zero in H . $T^\circ \subset E$ is $\sigma(H', H)$ -compact and equicontinuous, hence compact in E . Suppose $C \subset E$ is compact. Then C is $\sigma(H', H)$ bounded, hence equicontinuous since H is barrelled. So C° is a neighborhood of zero. So

$\exists \lambda > 0$ such that $\lambda T \subset C^\circ$ since T is bounded. So $(\lambda T)^\circ \supset C^{\circ\circ} \supset C$. But $(\lambda T)^\circ = \lambda^{-1} T^\circ$. So $T^\circ \supset \lambda C$. So T° absorbs C . So T° is a compact set absorbing all compact sets. Also $E = H^P$ is p -reflexive, because H is p -reflexive and the duals of p -reflexive things are p -reflexive.

2) (\Rightarrow) I claim E^P is a Banach space. Suppose R is a compact set which absorbs all compact sets. Then R° is a neighborhood of zero in E^P . Suppose C is any compact set of E . Then there exists a $\lambda > 0$ such that $\lambda C \subset R$. So $\lambda^{-1} C^\circ \supset R^\circ$. This implies that R° is bounded. Also E^P is p -reflexive since E is. So by 1), E^P is a Banach space. But since E is p -re-

flexive, $E \cong E^{PP}$. So let $H = E^P$. QED on 2)

And 3) follows from 2) by using 2) of 6.3. QED

Section 7 - Direct sums, quotients and subspaces of dF spaces.

As a general rule in the category \mathcal{C} of locally convex Hausdorff spaces and continuous linear maps, if a space $E \in \text{ob}\mathcal{C}$ has a particular property, then closed subspaces of E and Hausdorff quotients of E need not have that property. For example if E is complete and A is a closed subspace of E , then E/A need not be complete. Also if E is barrelled and A is a closed subspace of E , then A need not be barrelled. However for some special properties such as that of being a Banach space or being a Frechet space, it is the case that if E has the property and if A is a closed subspace of E , then both E/A and A have the property.

Perhaps then it is due to the intimate relationship between Banach and dB spaces, and between Frechet spaces and dF spaces, that both dF and dB spaces pass the property of being dF or dB respectively on to closed subspaces and Hausdorff quotients.

7.1 Theorem.

If E is a dF (respectively, dB) space and A is a closed subspace of E , then both A and E/A are again dF (respectively, dB) spaces.

Proof: Let E be a dF (respectively, dB) space. Then

by 6.3 (resp., 6.7) there exists a Frechet (resp., Banach) space F such that $F^{\mathbb{P}} = E$. Then

A° is a closed subspace of F , in particular A° is a Frechet (resp., Banach) space. So $A^{\circ} \in \text{ob}\mathcal{D}$. Also F/A° is a Frechet (resp., Banach) space and is thus in $\text{ob}\mathcal{D}$. Let $R: \mathcal{D} \rightarrow \mathcal{D}^{\text{op}}$ be the functor defined in 3.2. By 3.10, R sends limits to limits.

Consider the inclusion map $i: A^{\circ} \rightarrow F$. i is a kernel of the canonical projection from F to F/A° . So $R(i)$ is a cokernel in \mathcal{D} . Thus by 4.7, the canonical map $k: (F^{\mathbb{P}}/\ker t_i)^{\sim} \rightarrow (A^{\circ})^{\mathbb{P}}$ induced by t_i , is an isomorphism and

$$\begin{array}{ccc}
 F^{\mathbb{P}} & \xrightarrow{t_i} & (A^{\circ})^{\mathbb{P}} \\
 \downarrow a & & \uparrow k \\
 F^{\mathbb{P}}/\ker t_i & \xrightarrow{b} & (F^{\mathbb{P}}/\ker t_i)^{\sim}
 \end{array}
 \quad \text{commutes.}$$

Now t_i is surjective by the Hahn-Banach theorem. So $k \cdot b$ is surjective. This implies that b is surjective, since k is injective. But by 2.10, b is an isomorphism onto its range. So b is an isomorphism from $F^{\mathbb{P}}/\ker t_i$ onto $(F^{\mathbb{P}}/\ker t_i)^{\sim}$. So $c = k \cdot b$ is an isomorphism from $F^{\mathbb{P}}/\ker t_i$ onto $(A^{\circ})^{\mathbb{P}}$. Thus $F^{\mathbb{P}}/\ker t_i$ is a dF (resp., dB) space, since A° is a Frechet (resp., Banach) space.

But $\ker t_i = \{\phi \in F' : \phi \circ i = 0\} = A^{\circ\circ}$, and $A^{\circ\circ} = A$ by the bipolar theorem. So $F^{\mathbb{P}}/\ker t_i = E/A$ is a dF (resp., dB) space.

Now F/A° is a Frechet (resp., Banach) and the canonical map $j: F \rightarrow F/A^\circ$ is surjective and open, and every compact set in F/A° is an image of a compact set in F by the lemma on page 274 of [19]. So

$t_j: (F/A^\circ)^P \rightarrow F^P$ is an isomorphism onto its range.

But by essentially the same proof as that of the "claim" in the proof of theorem 5.1, we see that $A^{\circ\circ} =$ the range of t_j . However $A = A^{\circ\circ}$, by the bipolar theorem. Thus $(F/A^\circ)^P$ is isomorphic to A . Thus A is a dF (resp., dB) space. QED

7.2 Remark. Köthe in §31.5 of [27] mentions a DF space (in the sense of Grothendieck) which has a closed subspace which is not a DF space. Thus dF spaces seem to have better permanence properties than DF spaces.

7.3 Proposition. A countable direct sum of dF spaces is a dF space.

Proof: Let $\{E_i : i \in \omega\}$ be a countable collection of dF spaces. Now $\oplus E_i \cong (\oplus E_i)^{PP} \cong (\prod E_i^P)^P$ by 5.1. But each E_i^P is a Frechet space and the countable product of Frechet spaces is again Frechet. Hence $\oplus E_i$ is isomorphic to the p-dual of a Frechet space, and hence is a dF space. QED

Section 8 - "Hom"-ing with Frechet and dF spaces.

I prove in this section:

1) If E is Frechet and F is dF, then $\text{hom}_p(E, F)$ is a dF space; and

2) If E is dF and F is Frechet, then $\text{hom}_p(E, F)$ is a Frechet space.

Recall that the Banach-Dieudonne theorem says that if E is metrizable, then the topology of precompact convergence on E' is the finest topology on E' which induces on equicontinuous subsets of E' the same topology as the topology of pointwise convergence.

In the course of proving 1) above, I show that if E is metrizable and F is a dF space, then $\text{hom}_p(E, F)$ is a k -space. The Banach-Dieudonne theorem is shown to follow from this result, in the case $F = \mathbb{K}$.

Also in this section I prove that if E is either a Frechet space or a dF space, then E is nuclear iff E^p is nuclear.

8.1 Theorem. If E is a dF (resp., dB) space and F is a Frechet (resp., Banach) space, then $\text{hom}_p(E, F)$ is a Frechet (resp., Banach) space.

Proof: (dF-Frechet case) Since F has a countable base for the neighborhood system at zero and since E

is hemi-compact, $\text{hom}_p(E, F)$ has a countable base for the neighborhood system at zero. Hence $\text{hom}_p(E, F)$ is metrizable.

(dB-Banach case) Let C be a compact set of E which absorbs all compact sets and let U be a bounded neighborhood of zero in F . Then one can trivially verify that $\{T \in \text{hom}(E, F) : T(C) \subset U\}$ is a bounded neighborhood of zero in $\text{hom}_p(E, F)$. Hence $\text{hom}_p(E, F)$ is a normed space.

(both cases) We now must show that $\text{hom}_p(E, F)$ is complete. But $\text{hom}_p(E, F)$ is p -complete by 1.23, because E is p -determined and F is p -complete. So 1.9 insures that $\text{hom}_p(E, F)$ is p -reflexive, and thus by 6.3 it is a Frechet space and therefore complete. QED

8.2 Corollary. With the same hypothesis as 8.1, we may conclude that $\text{Hom}(E, F) = \text{hom}_p(E, F)$.

Proof: This is immediate from the definition of Hom , since Frechet spaces are p -determined. QED

8.3 Theorem. If E is a metrizable locally convex space and F is a dF space, then $P \subset \text{hom}_p(E, F)$ is pre-compact iff there exists a neighborhood U of zero in E and a compact subset K of F such that $P(U) \subset K$.

8.4 Corollary. Every continuous linear mapping from a metrizable locally convex space to a dF space is compact. QED

Proof of 8.3: (\Rightarrow) Note that E is p -determined by 1.9.

Also there exists a Frechet space H such that $F = H^{\mathbb{P}}$ by 6.3.

Let \mathfrak{X} equal all precompact sets of H . So we can regard continuous linear maps from E to $F = H^{\mathbb{P}}$ as \mathfrak{X} hypocontinuous bilinear forms on $E \times H$, and we can regard equicontinuous sets of maps from E to $H^{\mathbb{P}}$ as \mathfrak{X} equihypocontinuous sets of bilinear forms on $E \times H$. Let $P \subset \text{hom}_p(E, F)$ be precompact, then P is equicontinuous. So regarded as bilinear forms P is \mathfrak{X} -equihypocontinuous. The canonical bilinear form from $E \times \text{hom}_p(E, F)$ to F is uniformly continuous when restricted to $S \times P$ provided $S \subset E$ is precompact (cf. prop. 5, Chap. 3, §4, of [4]). So $P(S)$ is precompact in $H^{\mathbb{P}} = F$ for all S precompact in E . But H is p -determined, so $P(S)$ is equicontinuous as a subset of H' . This says that as bilinear forms P is \mathfrak{C} -equihypocontinuous, where \mathfrak{C} = all precompact sets of E . Hence P is \mathfrak{C} - \mathfrak{X} equihypocontinuous. Hence P is separately equicontinuous as bilinear forms on $E \times H$. Hence by proposition 10 of chapter 3, §4 of [4], P is equicontinuous as bilinear forms on $E \times H$. Hence there exist neighborhoods U and V of zero in E and H respectively such that $P(U)(V) \subset \{k \in \mathbb{K} : |k| \leq 1\}$, i.e. $P(U) \subset V^\circ \subset H^{\mathbb{P}} = F$. But V° is equicontinuous and $\sigma(H', H)$ -compact, hence compact in $H^{\mathbb{P}}$. So V° is compact in F . Let $K = V^\circ$. Then $P(U) \subset K$. QED on (\Rightarrow)

So we'll be done provided we can prove

8.3a Proposition. Let E and F be locally convex spaces. If $A \subset \text{hom}_p(E, F)$ and there exists a precompact set $P \subset F$ and a neighborhood of zero $U \subset E$ such that $A(U) \subset P$, then A is precompact in $\text{hom}_p(E, F)$.

Proof of 8.3a: Let $x \in E$. There exists a $\lambda > 0$ such that $\lambda x \in U$. So $A(\lambda x) \in P$. Hence $A(x) \subset \lambda^{-1}P$. Thus $A(x)$ is precompact. Suppose V is a neighborhood of zero in F . Then there exists an $\alpha > 0$ such that $\alpha P \subset V$. So $A(\alpha U) = \alpha A(U) \subset \alpha P \subset V$. But αU is a neighborhood of zero. So A is equicontinuous. So by Ascoli (cf. theorem 2, §2, of [8]), A is precompact in $\text{hom}_p(E, F)$.

QED on 8.3a

QED

What follows is a generalization of the Banach-Dieudonne theorem.

8.5 Theorem. Let E be a metrizable locally convex space and F be a dF space. Then $\text{hom}_p(E, F)$ is a k -space.

8.6 Corollary. Let E be a metrizable locally convex space and F be a dF space. Then the topology of precompact convergence is the finest topology on $\text{hom}(E, F)$ which gives to each equicontinuous, pointwise precompact subset of $\text{hom}(E, F)$ the same relative topology as the topology of pointwise convergence. (Here I really mean "the finest topology" and not "the finest locally convex topology").

8.7 Remark. If $F = \mathbb{K}$, i.e. the scalars, then every equicontinuous subset of $\text{hom}(E, F)$ is pointwise precompact. Hence the corollary in this case yields the classical Banach-Dieudonne theorem.

Proof of 8.6: For the purposes of the proof of this corollary, we will say that a topology \mathcal{J} on $\text{hom}(E, F)$ has property (*) provided that the relative topology induced on every equicontinuous, pointwise precompact subset of $\text{hom}(E, F)$ by \mathcal{J} is the same as that induced by the topology of pointwise convergence.

Claim: The topology of precompact convergence has property (*).

On uniformly equicontinuous sets, the uniformities of precompact convergence and pointwise convergence agree (cf. [8], theorem 1 of §2). QED on claim

Let \mathcal{J} be any topology on $\text{hom}(E, F)$ which satisfies property (*).

Claim: On compact subsets of $\text{hom}_p(E, F)$ the relative topologies induced by \mathcal{J} and $\text{hom}_p(E, F)$ agree.

Let K be a compact set in $\text{hom}_p(E, F)$. By 1.6, K is equicontinuous. It is also obviously pointwise precompact. Hence the relative topology on K induced by \mathcal{J} and by the topology of pointwise convergence agree. But since K is equicontinuous by theorem 1 of §2 of [8], the relative topologies on K induced by $\text{hom}_p(E, F)$ and by the topology of pointwise convergence agree.

QED on claim

Claim: \mathcal{J} is coarser than the topology of precompact convergence.

$\text{hom}_p(E, F)$ is a k -space by 8.5, and hence has the property that $\text{hom}_p(E, F)$ is finer than any topology on $\text{hom}(E, F)$ which gives to the compact subsets of $\text{hom}_p(E, F)$ the same relative topology as $\text{hom}_p(E, F)$ (cf. C13).

QED on claim

So I've proved that the topology of precompact convergence is the finest topology with property (*).

QED on 8.6

Proof of 8.5: Let \mathcal{O} = all compact subsets of $\text{hom}_p(E, F)$. Let \mathcal{J} equal the set of all $R \subset \text{hom}(E, F)$ such that $R \cap A$ is closed in $\text{hom}_p(E, F)$ for all $A \in \mathcal{O}$. Note that \mathcal{J} forms the closed sets for a topology on $\text{hom}(E, F)$. In fact $(\text{hom}(E, F), \mathcal{J})$ is the coreflection in the category of k -spaces of the space $\text{hom}_p(E, F)$ (cf. C3). In particular $\text{hom}_p(E, F)$ is a k -space iff $\text{hom}_p(E, F) = (\text{hom}(E, F), \mathcal{J})$. It is via this route that I shall prove that $\text{hom}_p(E, F)$ is a k -space. To this end, we will need the following:

Claim: The topology \mathcal{J} is translation invariant.

To prove the claim, I only need show that if $T \in \text{hom}(E, F)$, then the map f defined by $f(x) = T + x$ is \mathcal{J} -continuous. Thus let $T \in \text{hom}(E, F)$ and C be \mathcal{J} -closed. Now $f^{-1}(C) = \{x : T + x \in C\} = -T + C$. Let $A \in \mathcal{O}$. $A \cap (-T + C) = (T + A) \cap C$. But $T + A$ is compact in $\text{hom}_p(E, F)$. So since C is \mathcal{J} -closed,

$(T + A) \cap C = A \cap (-T + C)$ is $\text{hom}_p(E, F)$ closed. So $-T + C$ is \mathcal{J} -closed. Hence f is continuous. So \mathcal{J} is translation invariant. QED on claim

Since \mathcal{J} is translation invariant and the topology of $\text{hom}_p(E, F)$ is coarser than that of \mathcal{J} , all that we need to do in order to show that the two topologies are equal is

8.5a Claim: If W is \mathcal{J} -open and contains zero, then there exists a neighborhood X of zero in $\text{hom}_p(E, F)$ such that $X \subset W$.

Proof of claim: Let W be a \mathcal{J} -open set containing zero. Let H be a Frechet space such that $F = H^{\mathbb{P}}$ (cf. 6.3). Let $\{U_n\}$ and $\{V_n\}$ be countable bases for the neighborhood system at zero of E and H respectively. Assume $n \geq m$ implies that $U_n \subset U_m$ and $V_n \subset V_m$, and that $U_0 = E$ and $V_0 = H$.

Subclaim: For all integers $n \geq 0$, there exist finite sets $A_n \subset U_n$ and $B_n \subset V_n$ such that if one defines $C_n = \bigcup_{p=0}^n A_p$ and $D_n = \bigcup_{p=0}^n B_p$, then for all $n \geq 0$,

$\{T \in \text{hom}(E, H^{\mathbb{P}}) : T(U_{n+1}) \subset V_{n+1}^\circ \text{ and } T(C_n) \subset D_n^\circ\} \subset W$.

Proof of subclaim: Suppose $n \geq 0$, and A_0, \dots, A_{n-1} and B_0, \dots, B_{n-1} have been chosen which satisfy the conclusion. Suppose each element of the set

$$\mathcal{Q} = \left\{ \{R \in \mathcal{Q} : R(C_{n-1} \cup L) \subset (D_{n-1} \cup M)^\circ\} : \begin{array}{l} L \text{ and } M \text{ are finite} \\ \text{subsets of } U_n \text{ and} \\ V_n \text{ respectively} \end{array} \right\}$$

is non-empty, where

$Q = \{T \in \text{hom}(E, H^P) : T(U_{n+1}) \subset V_{n+1}^\circ\} \sim W$. Now V_{n+1}° is compact in $H^P = F$, so by 8.3 $\{T \in \text{hom}(E, F) : T(U_{n+1}) \subset V_{n+1}^\circ\}$ is precompact in $\text{hom}_p(E, F)$. Also V_{n+1}° is closed in $H^P = F$, so $\{T \in \text{hom}(E, F) : T(U_{n+1}) \subset V_{n+1}^\circ\}$ is closed in $\text{hom}_p(E, F)$. Since E is p -determined (it is metrizable) and $F = H^P$ is p -complete (it is dF), $\text{hom}_p(E, F)$ is p -complete by 1.23. Hence $\{T \in \text{hom}(E, F) : T(U_{n+1}) \subset V_{n+1}^\circ\}$ is compact in $\text{hom}_p(E, F)$. But by C4 we see that on the sets of \mathcal{O} , the topologies of precompact convergence and \mathcal{J} agree. So $\{T \in \text{hom}(E, F) : T(U_{n+1}) \subset V_{n+1}^\circ\}$ is \mathcal{J} -compact. Now W is \mathcal{J} -open. So Q is a closed subspace of a compact space and is hence \mathcal{J} -compact.

For every finite subset $L \subset U_n \subset E$ and $M \subset V_n \subset H$, $\{R \in \text{hom}(E, H^P) : R(C_{n-1} \cup L) \subset (D_{n-1} \cup M)^\circ\}$ is closed in $\text{hom}_p(E, F)$ and hence \mathcal{J} -closed, since $(D_{n-1} \cup M)^\circ$ is closed in $H^P = F$. So each of the elements of \mathcal{B} are \mathcal{J} -closed subsets of the \mathcal{J} -compact set Q .

Suppose L_i and M_i are finite subsets of U_n and V_n respectively for all i such that $1 \leq i \leq k$, for some finite ordinal k . Then

$$\bigcap_{i=1}^k \{R \in Q : R(C_{n-1} \cup L_i) \subset (D_{n-1} \cup M_i)^\circ\} \supset$$

$$\{R \in Q : R(C_{n-1} \cup \bigcup_{i=1}^k L_i) \subset (D_{n-1} \cup \bigcup_{i=1}^k M_i)^\circ\} \in \mathcal{B}.$$

Recalling that we are assuming each of the elements of \mathcal{B} to be non-empty, the last paragraph demonstrates that \mathcal{B} has the finite intersection property. Thus $\bigcap \mathcal{B} \neq \emptyset$ by compactness.

Suppose $z \in \cap \mathcal{Q}$. Then

subsubclaim: $z(U_n) \subset V_n^\circ$ and $z(C_{n-1}) \subset D_{n-1}^\circ$.

Letting $L = \{u\}$ and $M = \{v\}$ where $u \in U_n$ and $v \in V_n$, we see that $z \in \{R \in \mathcal{Q} : R(C_{n-1} \cup L) \subset (D_{n-1} \cup M)^\circ\}$, since $z \in \cap \mathcal{Q}$. So $|z(u)(v)| \leq 1$. Thus $z(U_n) \subset V_n^\circ$.

In order to see that $z(C_{n-1}) \subset D_{n-1}^\circ$, let $L = M = \emptyset$.

QED on subsubclaim

subsubclaim: $z \in W$.

Case 1. $n = 0$. In this case $z = 0$, since $U_0 = E$ and $V_0 = H$ which implies $V_0^\circ = \{0\}$. But $0 \in W$.

Case 2. $n \geq 1$. By induction hypothesis, $\{T \in \text{hom}(E, H^P) : T(U_n) \subset V_n^\circ \text{ and } T(C_{n-1}) \subset D_{n-1}^\circ\} \subset W$. So by the last subsubclaim $z \in W$. QED on subsubclaim

But $z \in \mathcal{Q} \subset \text{hom}(E, F) \sim W$, since $z \in \cap \mathcal{Q}$. Contradiction, since $z \in W$ and $z \notin W$. Thus our assumption that \mathcal{Q} does not contain the empty set is incorrect. Hence there exist finite sets L and M of U_n and V_n respectively such that $\emptyset = \{R \in \mathcal{Q} : R(C_{n-1} \cup L) \subset (D_{n-1} \cup M)^\circ\}$. This implies $\{R \in \text{hom}(E, H^P) : R(C_{n-1} \cup L) \subset (D_{n-1} \cup M)^\circ \text{ and } T(U_{n+1}) \subset V_{n+1}^\circ\} \subset W$. Letting $A_n = L$ and $B_n = M$, we have proved the induction step. QED on subclaim

Now to finish the proof of 8.5a, let $A = \bigcup_{n=0}^{\infty} A_n$ and $B = \bigcup_{n=0}^{\infty} B_n$. Recall that $\{U_n\}$ and $\{V_n\}$ are bases for the neighborhood filter at zero and that $n \geq m$ implies that $A_n \subset U_n \subset U_m$ and $B_n \subset V_n \subset V_m$. Hence

$A \cup \{0\}$ and $B \cup \{0\}$ are countable compact sets. Thus B° is a neighborhood of zero in $F = H^p$ and $\{T \in \text{hom}(E, F) : T(A) \subset B^\circ\}$ is a neighborhood of zero in $\text{hom}_p(E, F)$. Let $X = \{T \in \text{hom}(E, F) : T(A) \subset B^\circ\}$. I claim that $X \subset W$. To this end, let $T \in X$. By applying 8.4, we find that the map T is compact. Hence there exists a neighborhood of zero U in E and a compact set K in F such that $T(U) \subset K$. Now $F = H^p$ and H is p -determined, so K is equicontinuous. So K° is a neighborhood of zero in H . Choose $n \in \omega$ such that $U_{n+1} \subset U$ and $V_{n+1} \subset K^\circ$. So $V_{n+1}^\circ \supset K^{\circ\circ}$ and $K^{\circ\circ} \supset K$. Thus $T(U_{n+1}) \subset V_{n+1}^\circ$. Also $T(C_n) \subset D_n^\circ$ since $C_n \subset A$ and $D_n \subset B$ and $T(A) \subset B^\circ$. Hence $T \in W$ by the subclaim. So $X \subset W$. QED on 8.5a

Thus the topology \mathcal{J} equals the topology of $\text{hom}_p(E, F)$. Hence $\text{hom}_p(E, F)$ is a k -space. QED

8.8 Corollary. If E is metrizable and F is a dF space, then $\text{hom}_p(E, F)$ is a dF space.

Proof: $\text{hom}_p(E, F)$ is p -complete since F is p -complete and E is p -determined. If R is a locally convex space, then by 2.1 the topology of $\alpha(R)$ agrees with that of R on compact sets of R (where α is the p -determination functor), hence the finest such topology is finer than that of $\alpha(R)$, which in turn is finer than that of R . By the last theorem, $\text{hom}_p(E, F)$ is a k -space and hence the first and the last of these topo-

logies agree (cf. C13). Since $\alpha(\text{hom}_p(E,F))$ gets trapped in the middle, $\text{hom}_p(E,F)$ must be p -determined. So $\text{hom}_p(E,F)$ is p -reflexive.

Let $\{U_m\}$ and $\{K_n\}$ be countable cofinal sets in the set of neighborhoods of zero in E and the set of compact subsets of F respectively.

Let $Q_{mn} = \{T \in \text{hom}(E,F) : T(U_m) \subset K_n\}$. Each Q_{mn} is precompact by 8.3 and closed since each K_n is closed. Hence each Q_{mn} is compact.

Suppose $C \subset \text{hom}_p(E,F)$ is compact, then by 8.3 there exists a neighborhood U of zero in E and a compact set $K \subset F$ such that $C(U) \subset K$. Choose $U_m \subset U$ and $K_n \supset K$. Then $C \subset Q_{mn}$.

So $\{Q_{mn} : (m,n) \in \omega \times \omega\}$ is a countable cofinal set in the set of all compact sets of $\text{hom}_p(E,F)$. Hence $\text{hom}_p(E,F)$ is a dF space. QED

8.9 Corollary. If E is normable and F is a dB space, then $\text{hom}_p(E,F)$ is a dB space.

Proof: By corollary 8.8, $\text{hom}_p(E,F)$ is a dF space and hence is p -reflexive. Let U_0 be a bounded neighborhood of zero in E and let K_0 be a compact set of F which absorbs all compact sets. Let C be a compact set in $\text{hom}_p(E,F)$. Then by 8.3, there exists a compact set K in F and a neighborhood of zero U in E such that $C(U) \subset K$. Choose $\lambda > 0$ and $\eta > 0$ such that $\lambda U_0 \subset U$ and $\eta K \subset K_0$. So we have $\eta \lambda C \subset \{T \in \text{hom}(E,F) : T(U_0) \subset K_0\}$, since if $u \in U_0$ and

$c \in C$, then $\eta\lambda c(u) = \eta c(\lambda u) \in \eta c(U) \subset \eta K \subset K_0$.

Now $\{T \in \text{hom}(E, F) : T(U_0) \subset K_0\}$ is precompact by 8.3 and is closed since K_0 is closed. Hence it is compact. So it is a compact set which absorbs all compact sets. Thus $\text{hom}_p(E, F)$ is a dB space. QED

8.10 Corollary. If E is a metrizable locally convex space and F is a dF space, then $\text{Hom}(E, F) = \text{hom}_p(E, F)$.

Proof: $\text{hom}_p(E, F)$ is a dF space and is hence p-determined. So $\alpha(\text{hom}_p(E, F)) = \text{hom}_p(E, F)$. QED

The following proposition shows that nuclearity of Frechet and dF spaces is preserved by the functor $E \mapsto E^p$.

8.11 Proposition. Let E be a Frechet space. Then E is a nuclear space iff E^p is a nuclear space.

Proof: (\Rightarrow) Suppose E is nuclear, then bounded subsets of E are precompact. Hence the topology of bounded convergence on E' agrees with the topology of precompact convergence. Hence E^p is nuclear by a theorem of Grothendieck (cf. prop. 50.6 of [46]).

(\Leftarrow) [For this part of this proof I will use the notation and definitions of [46], e.g. \otimes will mean algebraic tensor product and $\hat{}$ will mean "the completion of".] Suppose E^p is nuclear. By theorem 50.1 (e) of [46], E is nuclear provided for every Banach space F , the canonical map of $E \hat{\otimes}_\pi F$ into $E \hat{\otimes}_E F$ is an iso-

morphism. Let F be a Banach space. Now this canonical map is continuous and $E \hat{\otimes}_{\pi} F$ and $E \hat{\otimes}_{\epsilon} F$ are both Frechet spaces (cf. Prop. 5, chap. 1 of [17] and prop. 9, §1, chap. 1 of [40]). So by the open mapping theorem it will be sufficient to show that the canonical map is both surjective and injective.

Recall that nuclear spaces satisfy the approximation property (cf. proposition 4, expose 17 of [42]).

8.11a Lemma. If E and F are Frechet spaces such that either E^p or F have the approximation property, then the canonical map from $E \hat{\otimes}_{\pi} F$ to $E \hat{\otimes}_{\epsilon} F$ is injective.

Proof: Recall that $E \hat{\otimes}_{\pi} F$ and $E \hat{\otimes}_{\epsilon} F$ are Frechet spaces, and are hence p -reflexive. Now the canonical map $b: E \hat{\otimes}_{\pi} F \rightarrow E \hat{\otimes}_{\epsilon} F$ will be injective provided $t_b: (E \hat{\otimes}_{\epsilon} F)^p \rightarrow (E \hat{\otimes}_{\pi} F)^p$ has a dense image. This is because ${}^{tt}b = b$, $(E \hat{\otimes}_{\epsilon} F)^{pp} = E \hat{\otimes}_{\epsilon} F$, and $(E \hat{\otimes}_{\pi} F)^{pp} = E \hat{\otimes}_{\pi} F$.

Claim: $(E \hat{\otimes}_{\pi} F)^p = B(E, F)$ where $B(E, F)$ denotes all continuous scalar-valued bilinear forms defined on $E \times F$ with the topology of uniform convergence on products of compact sets.

Proof of claim: Setwise, it is trivial that $(E \hat{\otimes}_{\pi} F)' = B(E, F)$ (cf. prop. 43.4 of [46]). Also since the canonical bilinear map from $E \times F$ to $E \hat{\otimes}_{\pi} F$ is continuous, it is clear that the map from $(E \hat{\otimes}_{\pi} F)^p$ to $B(E, F)$ is continuous. That the map from $B(E, F)$ to $(E \hat{\otimes}_{\pi} F)^p$ is continuous follows from the following fact:

if E and F are Frechet spaces and P is a precompact subset of $E \hat{\otimes}_{\pi} F$, then there are compact sets K in E and L in F such that P is contained in the closed, balanced, convex hull of $K \otimes L$ (cf. cor. 2 to theorem 45.2 of [46]). qed on claim

Claim: $E' \otimes F' \subset (E \hat{\otimes}_{\epsilon} F)'$.

Proof of claim: As sets $(E \hat{\otimes}_{\epsilon} F)' = (E \otimes_{\epsilon} F)'$.

So it will suffice to show that $E' \otimes F' \subset (E \otimes_{\epsilon} F)'$.

Now Treves in [46] proves that $E \otimes_{\epsilon} F$ can be identified with all continuous scalar-valued bilinear forms on

$E'_{\sigma} \times F'_{\sigma}$ with the topology of uniform convergence on products of equicontinuous sets (cf. 43.1 and prop. 42.4 of

[46]). Let $\phi \otimes \psi \in E' \otimes F'$. I would like to show that $\phi \otimes \psi \in (E \otimes_{\epsilon} F)'$. Suppose $b_{\alpha} \rightarrow 0$ in $B(E'_{\sigma}, F'_{\sigma})$.

Then $(\phi \otimes \psi)(b_{\alpha}) \rightarrow 0$ because $\{\phi\} \times \{\psi\}$ is a product of equicontinuous sets, so $b_{\alpha}(\phi, \psi) \rightarrow 0$. But

$(\phi \otimes \psi)(b_{\alpha}) = b_{\alpha}(\phi, \psi)$. QED on claim

So to show that the image of ${}^t b$ is dense in $(E \hat{\otimes}_{\pi} F)^{\mathcal{P}}$, it will suffice to show that the image of $E' \otimes F'$ under ${}^t b$ is dense in $B(E, F)$ where the latter space is as in the first claim.

Let \mathcal{C} and \mathcal{X} denote all compact subsets of E and F respectively. By the corollary to theorem 34.1 of [46], $B(E, F)$ equals the set of scalar-valued $\mathcal{C} - \mathcal{X}$ hypocontinuous bilinear forms on $E \times F$. So $B(E, F)$ can be identified with $L(F, E^{\mathcal{P}})$ where this space has the topology of uniform convergence on compact sets. Now by hypothesis, either $E^{\mathcal{P}}$ or F satisfy the

approximation property

Hence $F' \otimes E'$ is dense in $L(F, E^P)$ (cf. A_2 and A_3 of theorem 2I, exposé 14 of [42]).

Hence t_b has a dense image. So b is injective.

QED on lemma

Applying this lemma to the proposition we see that all that remains to be done is to prove that the canonical map from $E \hat{\otimes}_\pi F$ to $E \hat{\otimes}_\epsilon F$, which I shall continue to call b , is surjective.

Now $E \hat{\otimes}_\epsilon F \subset L(E^P, F)$ (cf. Prop. 5, exposé 8 of [42]). Let $L_1(E^P, F)$ denote all nuclear mappings from E^P to F . Since E^P is assumed nuclear, we have $L_1(E^P, F) = L(E^P, F)$. So $E \hat{\otimes}_\epsilon F \subset L_1(E^P, F)$. Let $\theta \in E \hat{\otimes}_\epsilon F$. Then $\theta \in L_1(E^P, F)$. So by prop. 47.2 of [46], there exist Banach spaces E_1 and F_1 and continuous maps $f: E^P \rightarrow E_1$, $v: E_1 \rightarrow F_1$, and $g: F_1 \rightarrow F$ such that v is nuclear and $\theta = g \circ v \circ f$. Now since E_1 and F_1 are Banach spaces and v is nuclear, v is in the image of the canonical map from $E_1^* \hat{\otimes}_\pi F_1$ to $L(E_1, F_1)$, where $(-)^*$ denotes the dual space with the topology of bounded convergence. Consider the diagram:

$$\begin{array}{ccc}
 E_1^* \hat{\otimes}_\pi F_1 & \xrightarrow{a} & L_1(E_1, F_1) \\
 \downarrow t_f \otimes g & & \downarrow q \\
 E^P \hat{\otimes}_\pi F & \xrightarrow{b} & L_1(E^P, F)
 \end{array}$$

where $q(z) = g \circ z \circ f$. By checking on elementary tensors,

it is easy to see that this diagram commutes. Thus by the comments above, there exists a $y \in E_1^* \hat{\otimes}_\pi F_1$ such that $a(y) = v$. Hence $q(a(y)) = q(v) = g \circ v \circ f = \theta$. Thus $b(({}^t f \otimes g)(y)) = \theta$. But E^P is a nuclear space and hence has the property that bounded sets are precompact. Hence $E^{P*} = E^{PP} \cong E$. So there exists an $x \in E \hat{\otimes}_\pi F$ such that $b(x) = \theta$ (take $x = ({}^t f \otimes g)(y)$). Thus b is surjective.

Hence b is an isomorphism. Thus E is a nuclear space. QED

8.12 Corollary. If E is either a Frechet space or a dF space, then E is nuclear iff E^P is nuclear.

Proof: The proposition just proved proves this theorem in the case when E is Frechet. If E is a dF space, then there exists a Frechet space H such that $E \cong H^P$, whence $E^P \cong H^{PP} \cong H$. Thus 8.11 handles this situation as well. QED

8.13 Corollary. If E is either a Banach space or a dB space, then E is nuclear iff E is finite dimensional.

Proof: The case when E is a Banach space is well-known. So suppose E is a dB space. Then E is a dF space. So if E is nuclear, then E^P is nuclear. But E^P is a Banach space. Thus E^P is finite dimensional. So $E \cong E^{PP}$ is finite dimensional. QED

Section 9 - Bilinear forms on products of dF spaces.

It is well-known that the following theorem is true for Frechet spaces. Perhaps then it is not surprizing that it is true for dF spaces.

9.1 Theorem. If E and F are dF spaces and G is any locally convex space, then any $\mathcal{C} - \mathcal{X}$ hypocontinuous bilinear form is continuous provided

1) $\mathcal{C} =$ all compact subsets of E and $\mathcal{X} = \{\{f\}: f \in F\}$,
or 2) $\mathcal{C} = \{\{e\}: e \in E\}$ and $\mathcal{X} =$ all compact subsets of F .

Proof: $E \oplus F$ is a dF space by 7.3. But by proposition 7 of chap. 2, §4, of [3], $E \oplus F \cong E \times F$. So $E \times F$ is a dF space. In particular $E \times F$ is a k-space by 6.4. Let $b: E \times F \rightarrow G$ be a $\mathcal{C} - \mathcal{X}$ hypocontinuous bilinear form. Let K be a compact subset of $E \times F$. Then there exist compact subsets K_1 and K_2 of E and F respectively such that $K \subset K_1 \times K_2$. By prop. 4 of chap. 3, §4, of [4], $b|_{K_1 \times K_2}$ and hence $b|_K$ is continuous. But $E \times F$ is a k-space, so b is continuous.

QED

For Frechet spaces, it is true that equihypocontinuous sets of bilinear forms are equicontinuous. I haven't been able to prove such a theorem for dF spaces.

Chapter Two

Topologically Free p -Reflexive Spaces

In this chapter, a functor M from the category of k -spaces and continuous maps to the category of p -reflexive spaces and continuous linear maps is defined in such a way that if X is a k -space, then there exists a continuous map $\epsilon: X \rightarrow M(X)$ with the following property: if E is a p -complete locally convex space and $f: X \rightarrow E$ is a continuous map, then there exists a unique continuous linear map $\bar{f}: M(X) \rightarrow E$ such that

$$\begin{array}{ccc}
 X & \xrightarrow{f} & E \\
 \downarrow \epsilon & \nearrow \bar{f} & \\
 M(X) & &
 \end{array}$$

commutes. For each k -space X , $M(X)$ may be thought of as being all compactly supported measures on X and $\epsilon: X \rightarrow M(X)$ may be thought of as being the function which assigns to each element x of X , the measure which is the point mass at x . And thus the linear function \bar{f} , described above, may be thought of as being defined by

$$\mu \longmapsto \int f \, d\mu .$$

Keep in mind, though, that $M(X)$ is defined for arbitrary k -spaces, not merely locally compact spaces.

The functor M is studied quite exhaustively in this chapter, with the bulk of the results appearing in section 14.

It should be noted that Bourbaki, in §3 of chapter 3 of [5], develops a theory which has much the same flavor

as mine (cf. proposition 9, §3, chapter 3 of [5]), although he leaves it in quite an embryonic state. It was Bourbaki's development which initially lead to my study.

Section 10 - An adjoint of the forgetful functor from the category of topological vector spaces to the category of topological spaces.

The existence of the adjoint is proved in 10.6. It is also shown that the adjunction can be lifted to a natural isomorphism in the category of p -reflexive spaces.

10.1 Conventions. Recall that $\alpha: \mathcal{C}_n \rightarrow \mathcal{O}_n$ denotes the coreflection functor from the category of locally convex spaces to the category of p -determined spaces.

For the remainder of this paper I shall make the following conventions:

If X is a topological space and E is a locally convex Hausdorff space, then $c(X, E)$ will denote all continuous functions from X to E ; $c_c(X, E)$ will denote $c(X, E)$ with the locally convex topology of uniform convergence on compact sets; and $C(X, E)$ will denote $\alpha(c_c(X, E))$. And if X is a topological space, then $c(X)$, $c_c(X)$, and $C(X)$ will denote $c(X, \mathbb{K})$, $c_c(X, \mathbb{K})$, and $C(X, \mathbb{K})$ respectively.

\mathcal{J} will denote the category of k -spaces and continuous maps; \mathcal{J}_0 will denote the category of Hausdorff topological spaces and continuous maps; and $k: \mathcal{J}_0 \rightarrow \mathcal{J}$ will denote the coreflection functor (see appendix C

for a discussion of k -spaces).

$J: \mathcal{C} \rightarrow \mathcal{J}_0$ will denote the functor which forgets all algebraic structure and $\gamma: \mathcal{C} \rightarrow \mathcal{J}$ will be defined by $\gamma = k \circ R$.

Note that in a natural way c is a functor from $\mathcal{J}_0^{\text{op}} \times \mathcal{C}$ to Sets; c_c is a functor from $\mathcal{J}_0^{\text{op}} \times \mathcal{C}$ to \mathcal{C} ; and C is a functor from $\mathcal{J}_0^{\text{op}} \times \mathcal{C}$ to \mathcal{O} .

The following results will be needed in the future.

10.2 Theorem. If X is a k -space and E is a p -complete uniform space, then $c_c(X, E)$ [i.e. all continuous functions from X to E with the uniformity of compact convergence] is a p -complete uniform space.

10.3 Corollary. If X is a k -space and E is a p -complete locally convex space, then $c_c(X, E)$ is a p -complete space and $C(X, E)$ is a p -reflexive space.

Proof of corollary: The only thing to prove is that if R is a p -complete space, then $\alpha(R)$ is also p -complete. But this is done in 2.15. QED on 10.3

Proof of theorem: Let X be a k -space and E be p -complete. Let \hat{E} be the completion of E . Regard E as a subspace of \hat{E} . Thus by 2.6f, E is p -closed in \hat{E} , i.e. if P is a precompact subset of E and P^- is the closure of P in \hat{E} , then $P^- \subset E$.

Now $c_c(X, \hat{E})$ is a complete uniform space (cf. theorem 7.12 of [21]) and the canonical map from $c_c(X, E)$

to $c_c(X, \hat{E})$ is uniformly continuous.

Let P be a closed precompact subset of $c_c(X, E)$. Then $P(t)$ is precompact in E for all $t \in X$. Hence for all $t \in X$, $P(t)^- \subset E$, where $P(t)^-$ is the closure of $P(t)$ in \hat{E} .

Let f_α be a cauchy net in P . Then f_α is cauchy in $c_c(X, \hat{E})$. So there exists a continuous function $g: X \rightarrow \hat{E}$ such that $f_\alpha \rightarrow g$ in $c_c(X, \hat{E})$. In particular $f_\alpha(t) \rightarrow g(t)$ for all $t \in X$. But for all $t \in X$, $f_\alpha(t) \in P(t)$. Hence for all $t \in X$, $g(t) \in P(t)^- \subset E$. So in fact g is a continuous function from X to E .

Let K be compact in X and let V be an element of the uniformity of E . Then there exists an element U of the uniformity of \hat{E} such that $(E \times E) \cap U = V$. Also there exists a β such that $\alpha \geq \beta$ implies that $(g(t), f_\alpha(t)) \in U$ for all $t \in K$. Thus if $\alpha \geq \beta$ and $t \in K$, then $(g(t), f_\alpha(t)) \in U \cap (E \times E) = V$. Hence $f_\alpha \rightarrow g$ in $c_c(X, E)$. But $f_\alpha \in P$ for all α , hence $g \in P$ since P is closed. Thus P is complete. Hence $c_c(X, E)$ is p -complete. QED

10.4 Corollary. c_c can be regarded as a functor from $\mathcal{J}^{\text{op}} \times \mathcal{B}$ to \mathcal{B} and C can be regarded as a functor from $\mathcal{J}^{\text{op}} \times \mathcal{B}$ to \mathcal{D} . QED

10.5 Definition. If X is a k -space, define $M(X)$ to be equal to $C(X)^P$ and $\varepsilon_X: X \rightarrow M(X)$ to be the evaluation map, i.e. $[\varepsilon_X(r)](f) = f(r)$. Note that some-

times the subscript will be omitted from the " ϵ_X ".

Now for the main theorem.

10.6 Theorem. 1) For all $X \in \mathcal{J}$, $M(X) \in \text{ob} \mathcal{D} \subset \text{ob} \mathcal{B}$; $\epsilon_X \in \text{Mor}_{\mathcal{J}_0}(X, \mathcal{J}(M(X)))$; the closed linear span of the image of ϵ_X equals $M(X)$; and if $f \in \text{Mor}_{\mathcal{J}_0}(X, \mathcal{J}(E))$ where $E \in \mathcal{B}$, then there exists a unique $\bar{f} \in \text{hom}(M(X), E)$ such that $\mathcal{J}(\bar{f}) \circ \epsilon_X = f$.

2) For all $X \in \mathcal{J}$, $\epsilon_X \in \text{Mor}_{\mathcal{J}}(X, \gamma(M(X)))$; and for all $X \in \mathcal{J}$ and $E \in \mathcal{B}$, if $f \in \text{Mor}_{\mathcal{J}}(X, \gamma(E))$, then there exists a unique $\bar{f} \in \text{hom}(M(X), E)$ such that $\gamma(\bar{f}) \circ \epsilon_X = f$.

3) a) If $X, Y \in \mathcal{J}$ and $f \in \text{Mor}_{\mathcal{J}}(X, Y)$ and $M(f)$ is defined to be the unique map from $M(X)$ to $M(Y)$ such that $\gamma(M(f)) \circ \epsilon_X = \epsilon_Y \circ f$; and b) if $E \in \mathcal{B}$ and μ_E is defined to be the unique map from $M(\gamma(E))$ to E such that $\gamma(\mu_E) \circ \epsilon_{\gamma(E)} = l_{\gamma(E)}$; then M is a functor from \mathcal{J} to \mathcal{B} , ϵ is a natural transformation from $\gamma \circ M$ to $I_{\mathcal{J}}$, μ is a natural transformation from $M \circ \gamma$ to $I_{\mathcal{B}}$, $\gamma(\mu_E) \circ \epsilon_{\gamma(E)} = l_{\gamma(E)}$ for all $E \in \mathcal{B}$, and $\mu_{M(X)} \circ M(\epsilon_X) = l_{M(X)}$ for all $X \in \mathcal{J}$. That is, μ and ϵ define an adjunction of M to γ , in particular M is an adjoint of γ .

4) The functors from $\mathcal{J}^{\text{op}} \times \mathcal{B} \rightarrow \mathcal{A}$ defined by $(X, E) \mapsto C(X, E)$ and $(X, E) \mapsto \text{Hom}(M(X), E)$ are naturally isomorphic.

Proof: Let X be a k -space and E be a p -complete

locally convex space. Let $b: c_c(X, E) \times E^{\mathcal{P}} \longrightarrow c_c(X)$ be the canonical bilinear form, i.e. $(f, \phi) \longmapsto \phi \circ f$. b is \mathcal{P} - \mathfrak{L} hypocontinuous, where $\mathcal{P} = \{\{f\} : f \in c(X, E)\}$ and $\mathfrak{L} =$ the set of equicontinuous subsets of E' . $E^{\mathcal{P}}$ is p -determined, since E is p -complete. Hence b is \mathcal{P} -hypocontinuous from $\alpha(c_c(X, E)) \times E^{\mathcal{P}}$ to $\alpha(c_c(X))$, by the universal property of α . I would like to show that b is \mathcal{P} - \mathfrak{L} hypocontinuous in this new setting. Let $L \in \mathfrak{L}$. $b(\cdot, L)$ is equicontinuous from $c_c(X, E)$ to $c_c(X)$, hence from $\alpha(c_c(X, E))$ to $c_c(X)$, since the topology of $\alpha(c_c(X, E))$ is finer than that of $c_c(X, E)$. Let $f \in \alpha(c_c(X, E))$. Now $b(f, L)$ is equicontinuous in $c(X)$ and $b(f, L)(x) = \{m(f(x)) : m \in L\}$ is a bounded, hence precompact, set of scalars for all $x \in X$. Hence $b(f, L)$ is precompact in $c_c(X)$ by Ascoli. Hence, again by Ascoli, $b(\cdot, L)$ is precompact in $\text{hom}_p(\alpha(c_c(X, E)), c_c(X))$. So by lemma 2.17, $b(\cdot, L)$ is equicontinuous from $\alpha(c_c(X, E))$ to $\alpha(c_c(X))$. Hence $b: C(X, E) \times E^{\mathcal{P}} \longrightarrow C(X)$ is \mathcal{P} - \mathfrak{L} hypocontinuous.

Define $\ell: C(X, E) \longrightarrow \text{hom}(E^{\mathcal{P}}, C(X))$ by $\ell(f) = b(f, \cdot)$. If $L \in \mathfrak{L}$ and $f \in C(X, E)$, $\ell(f)(L)$ is precompact in $C(X)$, since L equicontinuous implies L precompact in $E^{\mathcal{P}}$ (this is because on equicontinuous sets, the uniformities induced by $\sigma(E', E)$ and $E^{\mathcal{P}}$ agree). So if $f \in C(X, E)$, ${}^t[\ell(f)]$ is a continuous map from $[C(X)]^{\mathcal{P}}$ to $(E^{\mathcal{P}})'$ where the latter space has the topology of uniform convergence on equicontinuous sets of E' . Define $M = [C(X)]^{\mathcal{P}}$ and let $\delta: E \longrightarrow (E^{\mathcal{P}})'$ be the natural isomorphism (recall that E is p -complete).

10.6a Claim. M is p -reflexive.

Proof: M is p -complete because $C(X)$ is p -determined. M is p -determined because $c_c(X)$ is complete (since X is a k -space) and thus $C(X)$ is p -complete (cf. 2.15).
 Define $\varepsilon: X \rightarrow M$ by $\varepsilon(x)(f) = f(x)$. QED on claim

10.6b Claim. ε is continuous.

Proof: Let K be compact. I would like to show that $\varepsilon|_K$ is continuous. Let $x_0 \in K$. Let P be precompact in $C(X)$. Then P is also precompact in $c_c(X)$, hence $P|_K$ is equicontinuous. Let $\delta > 0$. Let U be a neighborhood of x_0 in K such that $x \in U$ implies that for all $f \in P$, $|f(x) - f(x_0)| < \delta$. Then $x \in U$ implies that for all $p \in P$, $|\varepsilon(x)(p) - \varepsilon(x_0)(p)| < \delta$. So $\varepsilon|_K$ is continuous at x_0 . Hence $\varepsilon|_K$ is continuous. But since X is a k -space, ε is continuous. QED on claim

Define $d: c(X, E) \rightarrow \text{hom}(M, E)$ by
 $d(f) = \delta^{-1} \circ {}^t[\varrho(f)]$, where $\delta: E \rightarrow (E^P)'$ is as above.

10.6c Claim. $d(f) \circ \varepsilon = f$, for all $f \in C(X, E)$.

Proof: Let $f \in C(X, E)$. If I can show that ${}^t[c(f)] \circ \varepsilon$ equals $\delta \circ f$, then I'm done. Let $\phi \in E'$ and let $x \in X$.
 $[{}^t[\varrho(f)](\varepsilon(x))](\phi) = \varepsilon(x)[\varrho(f)[\phi]] = (\phi \circ f)(x) = \phi(f(x))$.
 Also $[(\delta \circ f)(x)](\phi) = \phi(f(x))$. So they are equal.

QED on claim

10.6d Claim. The linear span of $\{\varepsilon(x) : x \in X\}$ is dense in M . Thus if $g: M \rightarrow E$ is continuous and linear and $g \circ \varepsilon = f$, then $g = d(f)$.

Proof: If $g \circ \varepsilon = f$, then $g \circ \varepsilon = d(f) \circ \varepsilon$ by 10.6c and thus $\varepsilon(x) \in \ker(d(f) - g)$ for all $x \in X$. So if

I can show that the span of ε is dense in M , then I will be done. Let $H =$ the closed linear span of $\{\varepsilon(x) : x \in X\}$. Suppose $\phi \in M'$ such that $\phi|_H = 0$. Now $M' = c(X)$ by the Mackey-Arens theorem, since $C(X)$ is p -complete. So there exists an $r \in c(X)$ such that for all $h \in M$, $\phi(h) = h(r)$. But $0 = \phi(\varepsilon(x)) = \varepsilon(x)(r) = r(x)$ for all $x \in X$. So $r = 0$. So $\phi = 0$. Hence $H = M$, by the Hahn-Banach theorem. QED on claim

Claim: $d: C(X,E) \longrightarrow \text{hom}_p(M,E)$ is continuous.

Proof: Let P be precompact in M and U be a closed, convex, balanced neighborhood of zero in E . U° is equicontinuous in E' . Since $C(X)$ is p -determined, P is equicontinuous from $C(X)$ to \mathbb{K} . So there exists a neighborhood V of zero in $C(X)$ such that $|p(v)| \leq 1$ for all $p \in P$ and $v \in V$. So since $b: C(X,E) \times E^P \longrightarrow C(X)$ is \mathfrak{L} -hypocontinuous, there exists a neighborhood W of zero in $C(X,E)$ such that $b(W \times U^\circ) \subset V$. I claim $d(W) \subset \{T \in \text{hom}(M,E) : T(P) \subset U\}$. Let $w \in W$ and $p \in P$. Is $d(w)(p) \in U$? Let $\phi \in U^\circ$. Then $|p(b(w,\phi))| \leq 1$. But $p(b(w,\phi)) = p(\ell(w)(\phi)) = (\tau[\ell(w)](p))(\phi)$. So $\tau[\ell(w)](p) \in U^{\circ\delta} \subset (E^P)'$. So $\delta^{-1} \circ \tau[\ell(w)](p) \in \delta^{-1}[U^{\circ\delta}] \subset U^{\circ\delta} \subset U$. But $\delta^{-1} \circ \tau[\ell(w)] = d(w)$. So $d(w)(p) \in U$. Hence $d: C(X,E) \longrightarrow \text{hom}_p(M,E)$ is a linear continuous map. QED on claim

Hence $d: C(X,E) \longrightarrow \alpha(\text{hom}_p(M,E)) = \text{Hom}(M,E)$ is continuous since $C(X,E)$ is p -determined.

Define $j: \text{hom}(M,E) \longrightarrow c(X,E)$ by $j(T) = T \circ \varepsilon$.

Note that $j \circ d = 1_{c(X,E)}$ by 10.6c and $d \circ j = 1_{\text{hom}(M,E)}$

by 10.6d. Note that j is continuous from $\text{hom}_p(M, E)$ to $c_c(X, E)$ because compact sets are precompact and because ε is continuous. Thus $j: \alpha(\text{hom}_p(M, E)) \longrightarrow \alpha(c_c(X, E))$ is continuous, i.e. $j: \text{Hom}(M, E) \longrightarrow C(X, E)$ is continuous. So both d and j are isomorphisms in the category \mathcal{C} .

Note that $\text{Hom}(M, E)$ is p -reflexive. This is because $\text{hom}_p(M, E)$ is p -complete, since M is p -determined and E is p -complete (cf. 2.15). Thus we have 10.6e Claim: $j: \text{Hom}(M, E) \longrightarrow C(X, E)$ defined by $T \longmapsto T \circ \varepsilon$ is a \mathcal{D} -isomorphism.

At this point we will begin calling M , $M(X)$ and begin calling ε , ε_X .

Let us now get down to proving 1) through 4) of the statement of theorem 10.6.

1) follows from 10.6a, 10.6b, 10.6c, and 10.6d.

2) Let $X \in \mathcal{J}$ and $E \in \mathcal{B}$. Realizing that $k(X)$ can be realized as living on the same set as X but with a finer topology, we see that the uniqueness and existence follow easily from 1). That ε_X is continuous from X to $k(J(E)) = Y(E)$ is evident from the fact that X is a k -space and k is a coreflection functor from \mathcal{J}_0 to \mathcal{J} .

3) follows from theorems 8.1 and 8.5 of [29] together with 2).

4) 10.6e shows that for all $X \in \mathcal{J}^{\text{op}}$ and $E \in \mathcal{B}$, $C(X, E)$ is isomorphic to $\text{Hom}(M(X), E)$ in the category \mathcal{D} . I must only show that the isomorphisms are "natural".

Suppose $(f, g) \in \text{Mor}_{\text{Top}_X \mathcal{B}}((X, E), (Y, F))$, i.e. $f \in \text{Mor}_J(Y, X)$ and $g \in \text{hom}(E, F)$. I want to show that

$$\begin{array}{ccc}
 \text{Hom}(M(X), E) & \xrightarrow{\text{Hom}(M(f), g)} & \text{Hom}(M(Y), F) \\
 \downarrow j(X, E) & & \downarrow j(Y, F) \\
 C(X, E) & \xrightarrow{C(f, g)} & C(Y, F)
 \end{array}$$

commutes, where j is as defined in 10.6e.

Let $T \in \text{Hom}(M(X), E)$. Then

$C(f, g)[j(X, E)(T)] = C(f, g)[T \circ \varepsilon_X] = g \circ (T \circ \varepsilon_X) \circ f$
 and $j(Y, F)[\text{Hom}(M(f), g)(T)] = j(Y, F)[g \circ T \circ M(f)] =$
 $(g \circ T \circ M(f)) \circ \varepsilon_Y$. But by the manner in which M is
 defined, we have $M(f) \circ \varepsilon_Y = \varepsilon_X \circ f$. Hence
 $j(Y, F)[\text{Hom}(M(f), g)(T)] = g \circ T \circ \varepsilon_X \circ f$. So
 $j(Y, F) \circ \text{Hom}(M(f), g) = C(f, g) \circ j(X, E)$ and thus the
 isomorphisms are natural. QED

The following proposition will aid in computations.

10.7 Proposition. If X is a k -space and $f: X \rightarrow \mathbb{K}$ is a continuous map, then $\mu \mapsto \mu(f)$ is the unique continuous linear map from $M(X)$ to \mathbb{K} such that

$$\begin{array}{ccc}
 X & \xrightarrow{f} & \mathbb{K} \\
 \varepsilon \downarrow & \nearrow & \\
 M(X) & &
 \end{array}$$

commutes.

Proof: Recall $M(X) = [C(X)]^{\mathbb{P}}$. Thus $\mu \mapsto \mu(f)$ is a continuous linear map doing the job. Uniqueness follows by 10.6. QED

10.8 Corollary. If X and Y are k -spaces, $f: X \rightarrow Y$ is continuous, $\mu \in M(X)$, and $g \in c(Y)$, then $[M(f)(\mu)](g) = \mu(g \circ f)$.

Proof: Let $\bar{g}: M(Y) \rightarrow K$ be the unique continuous linear map such that $\bar{g} \circ \varepsilon = g$. Consider

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & Y & \xrightarrow{g} & K \\
 \varepsilon \downarrow & & \downarrow \varepsilon & \nearrow [2] & \\
 & [1] & & & \bar{g} \\
 M(X) & \xrightarrow{M(f)} & M(Y) & &
 \end{array}$$

Diagram [1] commutes by the definition of $M(f)$ and diagram [2] commutes by the above comments. So the large diagram consisting of the outside arrows commutes. So $[\bar{g} \circ M(f)](\mu) = \mu(g \circ f)$ by 10.7. But we also have $\bar{g}[M(f)(\mu)] = [M(f)(\mu)](g)$ again by 10.7. Thus $[M(f)(\mu)](g) = \mu(g \circ f)$. QED

Section 11 - Simultaneous extension of continuous functions.

This section states, proves, and applies some results of Richard Arens and Ernest Michael to get continuous and in some cases continuous linear functions from $c_c(A, E)$ to $c_c(X, E)$ where A is closed in X and E is a topological vector space.

Also various minor consequences of these theorems are mentioned.

The results of this section will be of use in the study of the functor M a bit later.

11.1 Definition. A topological space is said to be Tychonoff provided it is completely regular and Hausdorff.

Recall that a topological space X is called hemi-compact if there exists a countable family \mathcal{K} of compact subsets of X such that for every compact subset A of X , there exists $K \in \mathcal{K}$ such that $A \subset K$.

11.2 Theorem. Let X be a Tychonoff and let A be a closed subset of X . Suppose either A is compact or X is hemi-compact. Let F be a Frechet space. Let $\psi: c_c(X, F) \rightarrow c_c(A, F)$ be the restriction map. Then

- 1) ψ is surjective, continuous, open, and linear;
- and 2) there exists a continuous map $\sigma: c_c(A, F) \rightarrow c_c(X, F)$

such that $\psi \circ \sigma = l_{c_c(A,F)}$.

11.3 Corollary. σ is a homeomorphism onto its range and the range of σ is closed.

Note that in both 11.2 and 11.3 σ need not be linear.

Proof of 11.2: Case 1): X is hemicompact.

Now X hemicompact $\Rightarrow X$ σ -compact $\Rightarrow X$ Lindelof $\Rightarrow X$ paracompact. Also X hemicompact and A closed implies A is hemicompact. So $c_c(X,F)$ and $c_c(A,F)$ are Frechet spaces. Hence $c_c(A,F)$ is paracompact. ψ is obviously continuous and linear. ψ is surjective by theorem 4.1 of [1]. Let U be a neighborhood of zero in $c_c(X,F)$. I claim $\psi(U)$ is a neighborhood of zero in $c_c(A,F)$. Let V be a closed, convex neighborhood of zero in F and $K \subset X$ be compact such that $\{g \in c(X,F) : g(k) \in V \text{ for all } k \in K\} \subset U$. Suppose $f \in c(A,F)$ such that $f(r) \in V$ for all $r \in A \cap K$. Define $g: A \cap K \rightarrow F$ by $g = f|_{A \cap K}$. By [1], there exists a function $p: K \rightarrow V$ such that $p|_{A \cap K} = g = f|_{A \cap K}$. Define a function $q: A \cup K \rightarrow F$ by $q|_K = p$ and $q|_A = f$. q is well-defined since $p|_{A \cap K} = f|_{A \cap K}$. Both K and A are closed in $A \cup K$ and $q|_A$ and $q|_K$ are continuous. So q is continuous; because if C is closed in F , then $q^{-1}(C) = (q|_K)^{-1}(C) \cup (q|_A)^{-1}(C)$. Thus by [1] again, extend q to a function $r: X \rightarrow F$ such that $r|_{A \cup K} = q$. Now $r(k) = p(k) \in V$ if $k \in K$.

So $r \in U$ and $\psi(r) = r|_A = f$. Hence $f \in \psi(U)$. Thus $\{f \in c(A, F) : f(k) \in V \text{ for all } k \in A \cap K\} \subset \psi(U)$. So $\psi(U)$ is a neighborhood of zero in $c_c(A, F)$.

Now suppose U is open in $c_c(X, F)$.

Claim: $\psi(U)$ is open in $c_c(A, F)$.

Proof: Let $f \in \psi(U)$. Suppose $p \in U$ such that $\psi(p) = f$. Then $-p + U$ is a neighborhood of zero. So by the above $\psi(-p + U)$ is a neighborhood of zero. But then $f + \psi(-p + U) \subset \psi(U)$, since $u \in U$ implies $f + \psi(-p + u) = f - \psi(p) + \psi(u) = \psi(u) \in \psi(U)$. So $\psi(U)$ is a neighborhood of f . Hence $\psi(U)$ is open.

QED on claim

Hence ψ is an open map.

Define $\phi: c(A, F) \rightarrow 2^{c(X, F)}$ by $\phi(f) = \psi^{-1}(\{f\})$.

For all $f \in c(A, F)$, $\phi(f)$ is closed, convex, and non-empty (since ϕ is surjective). Now ϕ is lower semi-continuous, since if V is open in $c_c(X, F)$, then $\{f \in c(A, F) : \phi(f) \cap V \neq \emptyset\} = \psi(V)$. Hence this set is open, because ψ is an open map.

Now $c_c(A, F)$ is paracompact and $c_c(X, F)$ is Frechet, so by [32], ϕ admits a continuous selection.

Hence there exists a continuous map $\sigma: c_c(A, F) \rightarrow c_c(X, F)$ such that $\psi \circ \sigma(f) = f$ for all $f \in c(A, F)$.

Case 2): A is compact.

Let $\beta(X)$ denote the Stone-Cech compactification of X . Since A is compact in X , it is closed in $\beta(X)$. Let $\nu: c_c(\beta(X), F) \rightarrow c_c(A, F)$ be the restriction map. Now applying case 1, we see that there exists a

continuous map $\tau: c_c(A, F) \rightarrow c_c(\beta(X), F)$ such that $v \circ \tau = 1_{c_c(A, F)}$. Let $\gamma: c_c(\beta(X), F) \rightarrow c_c(X, F)$ be the restriction map. It is continuous. Let U be an open set in $c_c(X, F)$ containing zero. Then $v(\gamma^{-1}(U))$ is open in $c_c(A, F)$, contains zero, and is contained in $\psi(U)$ (since $\psi \circ \gamma = v$). So ψ sends neighborhoods of zero to neighborhoods of zero. So ψ is open by the same argument as was used in case 1). ψ is trivially continuous and linear.

We know that $v \circ \tau = 1_{c_c(A, F)}$. So $\psi \circ (\gamma \circ \tau) = v \circ \tau = 1_{c_c(A, F)}$. Thus by letting $\sigma = \gamma \circ \tau$ we get that $\psi \circ \sigma = 1_{c_c(A, F)}$. QED

Proof of 11.3: Suppose f_α is a net in $c_c(A, F)$ and $\sigma(f_\alpha) \rightarrow \sigma(f_0)$. Then $f_\alpha = \psi(\sigma(f_\alpha)) \rightarrow \psi(\sigma(f_0)) = f_0$. So σ is a homeomorphism onto its range.

Now suppose $g \in c(X, F)$, f_α is a net in $c_c(A, F)$, and $\sigma(f_\alpha) \rightarrow g$. Then $f_\alpha = \psi(\sigma(f_\alpha)) \rightarrow \psi(g)$. So $\sigma(f_\alpha) \rightarrow \sigma(\psi(g))$. Hence $g = \sigma(\psi(g))$. So g is in the range of σ . Thus the range of σ is closed. QED

11.4 Theorem. Let E be a p -complete locally convex space with the property that compact subsets of E are metrizable. Let X be a Tychonoff space and K be a compact subset of X . Let $f: K \rightarrow E$ be a continuous function. Then there exists a continuous function $\bar{f}: X \rightarrow E$ such that $\bar{f}|_K = f$ and such that the range of \bar{f} is contained in the closed convex hull of the range of f .

Proof: Let $R =$ the closed convex hull of the range of f . R is precompact since the range of f is compact (prop. 3.9.7 of [19]). Since E is p -complete, R is compact, thus metrizable. So R is a convex, complete, metrizable space with the relative topology from E . Regarding K as a closed subspace of $\beta(X)$, by page 18 of [1] there exists a continuous function $g: \beta(X) \rightarrow R$ such that $g|_K = f$. Let $\bar{f} = g|_X$. Then the range of $\bar{f} \subset R$. QED

11.5 Corollary. Let E be a p -complete locally convex space with the property that compact subsets of E are metrizable. Let X be a Tychonoff space. Let Y be any Hausdorff topological space. If $r: Y \rightarrow X$ is continuous and injective, then $\psi: c_c(X, E) \rightarrow c_c(Y, E)$ defined by $\psi(g) = g \circ r$ has dense image.

Proof: Let U be a neighborhood of zero in E and $K \subset Y$ be compact. Let $f \in c(Y, E)$.

Now $r(K)$ is compact in X and $r|_K$ is a homeomorphism from K to $r(K)$ since it is injective and continuous.

Define $g: r(K) \rightarrow E$ by $g = f \circ (r|_K)^{-1}$. It is continuous, hence there exists a $\bar{g}: X \rightarrow E$ continuous such that $\bar{g}|_{r(K)} = g$ by the last theorem. Now $\psi(\bar{g})(k) = \bar{g}(r(k)) = g(r(k)) = f(r^{-1}(r(k))) = f(k)$ for all $k \in K$. So $\psi(\bar{g}) \in f + \{g \in c(Y, E) : g(k) \in U \forall k \in K\}$, since $(\psi(\bar{g}) - f)|_K = 0$. So the image of ψ is dense in $c_c(Y, E)$. QED

11.6 Corollary. If X and E are as in the statement of the last theorem, then

$\{f \in c(X,E) : (\text{Range } f)^- \text{ is compact}\}$ is dense in $c_c(X,E)$.

Proof: The set described is just the image of $c(\beta(X),E)$ under the canonical map induced by the inclusion map from X to $\beta(X)$, which is injective and continuous.

QED

The next two lemmas are needed in the proof of a theorem that follows.

11.7 Definition. Let \mathcal{E} denote the category of normed spaces and norm decreasing linear maps.

11.8 Lemma. Suppose E and $F \in \text{ob } \mathcal{E}$, $f \in \text{Mor}_{\mathcal{E}}(E,F)$, and $g \in \text{Mor}_{\mathcal{E}}(F,E)$. And suppose $g \circ f = 1_E$, then

- 1) f is an \mathcal{E} -isomorphism onto a closed linear subspace of F ;
- 2) g is an open surjective map; and
- 3) $F = (\text{Kernel } g) \oplus (\text{Image } f)$.

11.9 Lemma. Suppose E and $F \in \text{ob } \mathcal{E}$ and $E \neq \{0\}$. Let $f \in \text{Mor}_{\mathcal{E}}(E,F)$ and $g \in \text{Mor}_{\mathcal{E}}(F,E)$ be such that $g \circ f = 1_E$, then

- 1) f is an isometric isomorphism onto a closed linear subspace of F ;
- 2) g is an open surjective map such that $\|g\| = 1$;
- and 3) $f \circ g$ is a projection of norm one onto the image of f .

Proofs of 11.8 and 11.9: Both lemmas are quite trivial. 11.8 is basically a consequence of propositions 2 and 3 of chapter 2, section 7 of [19]. Some of the flavor of my 11.3 is also present.

As for 11.9, it basically follows from 11.8 together with a few observations.

Claim: f is an isometry.

Proof: $\|f(x)\| = 1 \cdot \|f(x)\| \geq \|g\| \|f(x)\| \geq \|g(f(x))\| = \|x\|$ because g is norm-decreasing. But f is norm-decreasing, so $\|f(x)\| \leq \|x\|$. Hence for all $x \in E$, $\|f(x)\| = \|x\|$. Thus f is an isometry onto its range.

QED on claim

Claim: g has norm one.

Proof: $\|g\| = \|g\| \cdot 1 \geq \|g\| \|f\| \geq \|g \circ f\| = \|1_E\| = 1$, since f is norm-decreasing and $E \neq \{0\}$. But by assumption g is norm-decreasing, so $\|g\| \leq 1$. Hence $\|g\| = 1$.

QED on claim

Claim: $f \circ g$ is a projection of norm one.

Proof: That $(f \circ g)^2 = f \circ g$ is trivial. $f \circ g$ is non-zero, since E is non-zero. All non-zero projections have norms greater than one. But $f \circ g$ is a morphism in \mathcal{E} , and is thus norm-decreasing. So $f \circ g$ must have norm equal to one.

QED on claim

QED

11.10 Theorem. (from [33]) Suppose X is a Tychonoff space and A is a closed subspace of X which is a metric space. Let E be a Hausdorff locally convex

topological vector space. Suppose that at least one of the following three conditions holds:

a) X is paracompact; and A is a G_δ or A is complete;

b) X is normal; A is separable; and A is a G_δ or A is complete; or

c) A is compact.

Let $r: c_c(X, E) \rightarrow c_c(A, E)$ be the restriction map. Then there exists a map $\phi: c_c(A, E) \rightarrow c_c(X, E)$ such that

1) ϕ is a \mathcal{C} -isomorphism onto a closed subspace of $c_c(X, E)$;

2) $r \circ \phi = 1_{c_c(A, E)}$;

3) for all $f \in c_c(A, E)$, $\text{Range } \phi(f) \subset \text{convex hull of Range } f$; and

4) $c_c(X, E) = \text{Image } \phi \oplus \text{Kernel } r$.

Note that Michael asserts that in a) and b) above, the condition that A be complete or A be a G_δ in X is not needed. However I am not able to construct a proof without these hypotheses.

Proof: Let $\beta(X)$ be the Stone-Cech compactification of X . Regard X and A as being subspaces of $\beta(X)$. If conditions a) or b) are satisfied, define $Y = X$; and if condition c) is satisfied, let $Y = \beta(X)$. Now A is a closed subspace of Y which is metric. Now either A is complete, or A is a G_δ in Y . Also Y is either paracompact, or Y is normal and A is separable. Applying lemma 4.3 of [33], we see that there exists a

metric space F which contains A as a closed subspace and a continuous function $R: Y \rightarrow F$ such that $R(a) = a$ for all $a \in A$. So if $\nu: c_c(F, E) \rightarrow c_c(A, E)$ is the restriction map, then by theorem 7.1 of [33] there exists a map $\psi: c_c(A, E) \rightarrow c_c(F, E)$ which satisfies conditions 2) and 3) of the conclusion.

Define $\gamma: c_c(F, E) \rightarrow c_c(Y, E)$ by $\gamma(g) = g \circ R$ and let $\delta: c_c(Y, E) \rightarrow c_c(A, E)$ be the restriction map. Both γ and δ are continuous and linear; and we have $\delta \circ \gamma = \nu$ because if $g \in c(F, E)$ and $a \in A$, then $\delta(\gamma(g))(a) = \delta(g \circ R)(a) = (g \circ R)(a) = g(R(a)) = g(a) = \nu(g)(a)$. So $\delta \circ (\gamma \circ \psi) = (\delta \circ \gamma) \circ \psi = \nu \circ \psi = 1_{c_c(A, E)}$ and $\gamma \circ \psi: c_c(A, E) \rightarrow c_c(Y, E)$ is continuous and linear.

Let $f \in c(A, E)$ and $y \in Y$. Then $(\gamma \circ \psi)(f)(y) = \psi(f)(R(y)) \in \text{Range}(\psi(f)) \subset \text{convex hull}(\text{Range } f)$. So $\text{Range } (\gamma \circ \psi)(f) \subset \text{convex hull}(\text{Range } f)$ for all $f \in c(A, E)$.

If condition c) is satisfied, then let $\lambda: c_c(\beta(X), E) \rightarrow c_c(X, E)$ be the restriction map. λ is continuous and linear. Recall $Y = \beta(X)$. Then $r \circ \lambda = \delta$ (r was defined in the statement of the theorem). Now $\lambda \circ \gamma \circ \psi: c_c(A, E) \rightarrow c_c(X, E)$ is continuous and linear, and $r \circ (\lambda \circ \gamma \circ \psi) = (r \circ \lambda) \circ (\gamma \circ \psi) = \delta \circ (\gamma \circ \psi) = (\delta \circ \gamma) \circ \psi = \nu \circ \psi = 1_{c_c(A, E)}$. Also if $\varrho \in c(\beta(X), E)$, then $\text{Range } \lambda(\varrho) \subset \text{Range } \varrho$. Thus if $f \in c(A, E)$, then $\text{Range } (\lambda \circ \gamma \circ \psi)(f) \subset \text{Range } (\gamma \circ \psi)(f) \subset \text{convex hull}(\text{Range } f)$.

If condition a) or b) are satisfied, define $\phi = \gamma \circ \psi$. Notice $r = \delta$ and $X = Y$, in this case.

If condition c) is satisfied, define $\phi = \lambda \circ \gamma \circ \psi$.

In each case we have $\phi: c_c(A, E) \longrightarrow c_c(X, E)$ such that $r \circ \phi = 1_{c_c(A, E)}$, ϕ is continuous and linear, and $\text{Range } \phi(f) \subset \text{convex hull}(\text{Range } f)$ for all $f \in c(A, E)$.

These conditions imply conditions 1), 2), 3), and 4) of the conclusion due to 11.8. QED

11.11 Corollary. If X is a metric space, A is a closed subspace of X , and E is a locally convex Hausdorff space, then the conclusion of theorem 11.10 is true.

Proof: X metric implies X is Tychonoff. Also X metric implies that condition a) of the last theorem holds. This is true because metric implies paracompact, and a closed subspace of a metric space is a G_δ . QED

The following shows that theorem 11.10 cannot be generalized too far.

11.12 Theorem. Let N equal the natural numbers with the discrete topology. Let $\beta(N)$ denote the Stone-Cech compactification of N . Let $A = \beta(N) \sim N$ and $X = \beta(N)$. Then A is closed in X and there exists no continuous linear map $\sigma: c_c(A) \longrightarrow c_c(X)$ such that $[\sigma(f)]|_A = f$ for all $f \in c(A)$.

Proof: Recall that if X is a locally compact Hausdorff space and $A \subset X$ is a dense subspace, then A is open

iff A is locally compact (cf. p.45 of [16]). This implies that every locally compact Hausdorff space is open when considered as a subspace of its Stone-Cech compactification.

Thus $\beta(N) \sim N$ is closed in $\beta(N)$.

Now by 11.8, if a map such as σ existed, then we would have $\ell_\infty = c_c(X) = \{f : f|_A = 0\} \oplus \sigma(c_c(A))$. But $\{f \in c(X) : f|_A = 0\} = c_0$. So c_0 would be complemented in ℓ_∞ , which is a contradiction (cf. [49]). QED

Section 12 - Topologically p-complete spaces

This section develops topologically p-complete spaces, a concept which will be very useful in what follows. These spaces are analogous to the concept of p-completeness for locally convex spaces.

Closely tied to the notion of topological p-completeness are two other concepts, relative topological precompactness and relative topological completeness. These concepts are studied as well.

12.1 Definition. Let X be a completely regular topological space. Let $A \subset X$. A will be said to be topologically precompact in X provided A is precompact in (X, \mathcal{U}) for every uniformity \mathcal{U} on X compatible with the topology. X will be called topologically precompact provided X is topologically precompact in X .

12.2 Proposition. If X is completely regular and A is topologically precompact in X , then the closure A^- of A in X is also topologically precompact in X .

Proof: In a uniform space, the closure of a precompact set is again precompact. QED

12.3 Proposition. 1) Let X be completely regular. Then

X is topologically precompact iff for all $A \subset X$, A is topologically precompact in X .

2) Let A be completely regular. Then A is topologically precompact iff for all X such that X is completely regular and $A \subset X$, A is topologically precompact in X .

3) Let X be completely regular. Suppose $A \subset B \subset X$. Then if A is topologically precompact in B , it follows that A is topologically precompact in X .

Proof: 1) and 2) are trivial. As for 3), suppose that \mathcal{U} is a uniformity of X compatible with the topology. Then $\mathcal{U}|_B$ is a uniformity on B compatible with the topology of B . Hence A is precompact for $(\mathcal{U}|_B)|_A$. But $(\mathcal{U}|_B)|_A = \mathcal{U}|_A$. So A is precompact in \mathcal{U} . QED

12.4 Note. There exist examples of spaces A and X where A is topologically precompact in X , but A is not topologically precompact. For example if $X = [0,1]$ and $A = (0,1) \subset \mathbb{R}$ and $f: A \rightarrow \mathbb{R}$ is defined by $f(x) = 1/x$, then we will see that A is not topologically precompact by the next theorem. But $[0,1]$ is topologically precompact. So $(0,1)$ is topologically precompact in $[0,1]$.

12.5 Definition. A collection of subsets \mathcal{E} of a topological space X is said to be locally finite (respectively, discrete) if for every $x \in X$, there exists a neighborhood of x which intersects at most

finitely many (respectively, one) of the sets of \mathcal{C} .

12.6 Theorem. Let X be a completely regular topological space. Let $A \subset X$. Then the following four conditions are equivalent (cf. 12.33):

- 1) A is topologically precompact in X .
- 2) If \mathcal{C} is a locally finite collection of open sets, then $\{C \in \mathcal{C} : C \cap A \neq \emptyset\}$ is finite.
- 3) For all f , if $f: X \rightarrow \mathbb{R}$ is continuous, then $f(A)$ is bounded.
- 4) If \mathcal{C} is a discrete collection of open sets, then $\{C \in \mathcal{C} : C \cap A \neq \emptyset\}$ is finite.

Proof: 1) \Rightarrow 3). Let $c(X)$ denote all continuous real-valued functions on X . Let \mathcal{U} be the coarsest uniformity on X making each element of $c(X)$ uniformly continuous. By the corollary to proposition 4 of §2, chap. 2, of [6], the topology \mathcal{J}^* induced by \mathcal{U} is the coarsest topology on X making each element of $c(X)$ continuous. So $\mathcal{J}^* \subset \mathcal{J}_0$, where \mathcal{J}_0 is the original topology on X . Suppose $x_\alpha \rightarrow x_0$ in \mathcal{J}^* , then $f(x_\alpha) \rightarrow f(x_0)$ for all $f \in c(X)$, since each element of $c(X)$ is continuous for \mathcal{J}^* (since they are uniformly continuous for (X, \mathcal{U})). Let U be an \mathcal{J}_0 -open neighborhood of x_0 . Because \mathcal{J}_0 is completely regular, there exists a $g: X \rightarrow \mathbb{R}$ continuous such that $g(x_0) = 1$ and $g(x) = 0$ for all $x \notin U$. Choose β such that $\alpha \geq \beta$ implies $|g(x_\alpha) - g(x_0)| < 1/2$. Then I claim that $\alpha \geq \beta$ implies $x_\alpha \in U$, for suppose that there

exists a $\alpha \geq \beta$ such that $x_\alpha \notin U$. Then $g(x_\alpha) = 0$. So $1 = |g(x_\alpha) - g(x_0)| < 1/2$. Contradiction. So $\alpha \geq \beta$ implies $x_\alpha \in U$. Hence $x_\alpha \rightarrow x_0$ in \mathcal{J}_0 . Hence $\mathcal{J}^* \supset \mathcal{J}_0$. Thus $\mathcal{J}^* = \mathcal{J}_0$. Hence \mathcal{U} is a uniformity compatible with the topology on X . So if A is topologically precompact in X , then A is precompact in (X, \mathcal{U}) . So if $f \in c(X)$, then $f(A)$ is precompact in \mathbb{R} , hence bounded, since each $f \in c(X)$ is uniformly continuous for \mathcal{U} . QED on $1 \Rightarrow 3$

3) \Rightarrow 2). Suppose \mathcal{C} is a locally finite collection of open subsets of X such that $\{C \in \mathcal{C} : C \cap A \neq \emptyset\}$ is infinite. Select a countably infinite subcollection $\{C_i : i \in \omega\}$ of \mathcal{C} such that $C_i \cap A \neq \emptyset$ for all $i \in \omega$ and such that $i \neq j$ implies $C_i \neq C_j$. Thus $\{C_i : i \in \omega\}$ is a locally finite open collection.

For all $i \in \omega$, choose an element of $C_i \cap A$ and call it a_i . Since X is completely regular, there exists a function $f_i : X \rightarrow [0, 1]$ such that $f_i(a_i) = 1$ and $f_i(x) = 0$ if $x \notin C_i$. Since locally finite implies point finite, we can define a function $g : X \rightarrow [0, \infty)$ by $g(x) = \sum_{i=1}^{\infty} i f_i(x)$. Note $g(a_j) = \sum_{i=1}^{\infty} i f_i(a_j) \geq j f_j(a_j) = j$ for all $j \in \omega$. So since $\{a_i : i \in \omega\} \subset A$, $g(A)$ is unbounded. Thus if we can show that g is continuous we are done.

Let $x \in X$. Let V be an open neighborhood of x such that $\{i \in \omega : C_i \cap V \neq \emptyset\}$ is finite. Then $g|_V = \sum \{i f_i|_V : i \in \{j \in \omega : C_j \cap V \neq \emptyset\}\}$, i.e. a finite linear combination of functions which are continuous on

on V . Hence $g|_V$ is continuous. Since V is open and contains x , this implies g is continuous at x .

Hence g is continuous.

QED on $3 \Rightarrow 2$

2) \Rightarrow 4). Trivial.

4) \Rightarrow 1). Suppose there exists a uniformity \mathcal{U} on X compatible with the topology such that A is not precompact in (X, \mathcal{U}) . Then there exists a $U \in \mathcal{U}$ such that $A \not\subset U[F]$ for all finite subsets F of A . Let a_1 be an arbitrary element of A . Inductively choose a sequence $\{a_i\}_{i=1}^{\infty} \subset A$ such that $a_{i+1} \notin U[a_1, \dots, a_i]$. Choose a symmetric element V of \mathcal{U} such that $V \circ V \subset U$.

Now suppose $r \in V[a_i] \cap V[a_j]$ and $i \neq j$. Suppose $i > j$. Then $(a_i, r) \in V$ and $(a_j, r) \in V$. Since V is symmetric, $(a_j, a_i) \in U$ and so $a_i \in U[a_j] \subset U[a_1, \dots, a_{i-1}]$. Contradiction. Thus $V[a_i] \cap V[a_j] \neq \emptyset$ implies $i = j$.

Choose $T \in \mathcal{U}$, symmetric, so that $T \circ T \subset V$.

Claim: $\{(T[a_i])^\circ : i \in \omega\}$ is a discrete collection of open sets. Note that I am using $^\circ$ to denote the interior operation.

Proof: Let $x \in X$. Then $(T[x])^\circ$ is an open set containing x which intersects at most one element of $\{(T[a_i])^\circ : i \in \omega\}$. Because if $\emptyset \neq (T[a_i])^\circ \cap (T[x])^\circ \subset T[a_i] \cap T[x]$, then this implies that $x \in (T \circ T)[a_i] \subset V[a_i]$. But the collection of $V[a_i]$ are disjoint. So $(T[x])^\circ$ intersects at most one $(T[a_i])^\circ$. Hence $\{(T[a_i])^\circ : i \in \omega\}$ is discrete. QED on claim

Now $(T[a_i])^\circ \cap A \neq \emptyset$, since $a_i \in A$. Also $i \neq j$ implies $(T[a_i])^\circ \neq (T[a_j])^\circ$, since for all $i \in \omega$,

$T[a_i] \subset V[a_i]$ and the $V[a_i]$ are disjoint. So there exists a countably infinite discrete collection of open sets each of which intersects A . QED on 4 \Rightarrow 1

12.7 Definition. A topological space is countably compact iff every countable open cover has a finite subcover.

12.8 Proposition. If X is a Tychonoff space, then X countably compact implies X is topologically precompact.

Proof: If X is countably compact and $f: X \rightarrow \mathbb{R}$ is continuous, then $f(X)$ is countably compact in \mathbb{R} , hence compact (cf. 5.5 and 5E of [21]). So X is topologically precompact by 12.6. QED

12.9 Proposition. If X is Hausdorff and normal, then X is topologically precompact iff X is countably compact.

Proof: §9 of Chapter 3 of [15] proves that Hausdorff, normal, topologically precompact spaces are countably compact. 12.8 proves the other direction. QED

12.10 Definition. Let X be a completely regular topological space. Let $A \subset X$. A will be said to be topologically complete in X provided there exists a uniformity \mathcal{U} on X compatible with the topology, such that A is a complete subspace of (X, \mathcal{U}) . X will be called topologically complete if X is topologically

complete in X .

12.11 Proposition. Let X be completely regular.

a) If X is topologically complete and A is closed in X , then A is topologically complete in X .

b) If A is topologically complete in X , then A is topologically complete. QED

12.12 Note. If X is completely regular and $A \subset X$, then A can be topologically complete at the same time that A is not topologically complete in X . For example, if $A = (0,1)$ and $X = [0,1]$, then A is topologically complete, but A is not topologically complete in X . Since X is compact, there is only one uniformity on X , and A with this relative uniformity is not complete. But A is metrizable, hence paracompact, hence topologically complete (cf. 12.14).

12.13 Proposition. If X is a completely regular space and $A \subset X$, then the following conditions are equivalent:

1) A is compact.

2) A is topologically complete in X and A is topologically precompact in X .

3) A is topologically complete and A is topologically precompact. QED

12.14 Theorem. (Problem 6L of [21]) Let X be a paracompact space. Let \mathcal{V} = all neighborhoods in the product topology of the diagonal. Then (X, \mathcal{V}) is a complete

uniform space compatible with the topology.

Proof: Using Kelley's terminology, the proof basically is a consequence of the fact that every open cover of a paracompact space is even. QED

12.15 Corollary. Paracompact spaces are topologically complete. QED

12.16 Corollary. Locally compact topological groups and pseudometric spaces are topologically complete.

Proof: Both such spaces are paracompact. QED

12.17 Real-compactness. This concept was introduced by E. Hewitt. One of the many equivalent forms of the definition is as follows: a completely regular space X is real-compact provided X is homeomorphic to a closed subset of a Cartesian product of real lines. See [15] for a discussion of real-compactness.

12.18 Proposition. If X is completely regular, then X real-compact implies X is topologically complete.

Proof: Any product of real lines is complete with the product uniformity, and closed subspaces of complete spaces are complete. Thus using 12.17, we are done. QED

12.19 through 12.26 have been omitted.

The following theorem will lead into the concept of topological p -completeness.

12.27 Theorem. Let X be a completely regular topological space. Then the following two conditions are equivalent:

1) There exists a uniformity \mathcal{U} on X compatible with the topology such that if A is any closed, precompact subset in (X, \mathcal{U}) , then A is compact.

2) If A is a closed subset in X which is topologically precompact in X , then A is compact.

Proof: 1) \Rightarrow 2). Trivial.

2) \Rightarrow 1). By problem 6G(b) in [21] and problem 5 of §1 of [7], there exists a uniformity \mathcal{U} on X which is the finest uniformity compatible with the topology. Suppose A is closed and \mathcal{U} -precompact, then A is precompact in any other uniformity on X compatible with the topology (since \mathcal{U} is finer than all of these). Hence A is topologically precompact in X . Hence A is compact. QED

12.28 Definition. A topological space X will be called topologically p-complete provided X is a Tychonoff and X satisfies one of the equivalent conditions of theorem 12.27.

12.29 Proposition. If X is topologically complete, then X is topologically p-complete. The converse is in general false.

Proof: Suppose A is closed and topologically precompact in X . Let \mathcal{U} be a uniformity on X compatible with the topology such that

(X, \mathcal{U}) is complete, then A is a complete subspace of (X, \mathcal{U}) , since it is closed. Since A is topologically precompact in X , A is precompact in this relative uniformity. Hence A is compact.

See [43] for an example of a topologically p -complete space which is not topologically complete. QED

12.30 Proposition. A closed subspace of a topologically p -complete space is topologically p -complete.

Proof: Let X be topologically p -complete and A closed in X . Let C be a closed subspace of A which is topologically precompact in A . Then C is closed in X and also topologically precompact in X (cf. 12.3). Hence A is compact. QED

12.31 Remark. Metrizable spaces, paracompact spaces, real-compact spaces, compact spaces, and locally compact groups are all topologically complete and hence are topologically p -complete. However there do exist non-topological p -complete spaces. We can get some of them in the following manner, including the first uncountable ordinal.

12.32 Proposition. If X is a completely regular, countably compact space which is not compact, then X is not topologically p -complete.

Proof: X is closed in itself and is topologically precompact in X (cf. 12.8). So if X were topologically

p -complete, X would be compact, but it is not. QED

The following should have been included in theorem 12.6. Note that the proof of it can be used to prove that 1 implies 3 in 12.6.

12.33 Theorem. Let X be a completely regular topological space. Let $A \subset X$. Then the following are equivalent:

- 1) A is topologically precompact in X .
- 2) If Y is a uniform space and $f: X \rightarrow Y$ is continuous, then $f(A)$ is precompact in Y .

Proof: By problem 5 of §1 of [7] and problem 6G (b) of [21], there exists a uniformity \mathcal{U} on X compatible with the topology of X such that if Z is any uniform space and $g: X \rightarrow Z$ is continuous, then $g: (X, \mathcal{U}) \rightarrow Z$ is uniformly continuous.

Assume A is topologically precompact in X . Then A is \mathcal{U} -precompact. Let Y be a uniform space and let $f: X \rightarrow Y$ be continuous. Then $f: (X, \mathcal{U}) \rightarrow Y$ is uniformly continuous. So $f(A)$ is precompact in Y . Thus we have proved that 1 implies 2.

2 implies 1 follows from 12.6. QED

12.34 Remark. The existence of the uniformity described in the proof of 12.33, demonstrates that the forgetful functor from uniform spaces (and uniformly continuous maps) to completely regular spaces (and continuous maps) has an adjoint.

Section 13 - An important consequence of a space being topologically p -complete.

The major theorem of this section (corollary 13.14) states that if E is a Frechet space and X is a topologically p -complete space, then $c_c(X,E)$ is barrelled.

The preliminary material in this section is well-known and is included only for the sake of completeness.

The major results of this section make more transparent, and generalize, theorems of Shirota [43] and of Nachbin [34].

13.1 Theorem. Let X be a Tychonoff space. Suppose K is a compact subset of X and suppose $\{V_1, \dots, V_n\}$ is a finite open cover of K . Then for each i , $0 \leq i \leq n$, there exists a $\phi_i: X \rightarrow [0,1]$, continuous, such that $\text{Supp } \phi_0 \subset X \sim K$, for all i with $1 \leq i \leq n$, $\text{Supp } \phi_i \subset V_i$, and $\sum_{i=0}^n \phi_i(x) = 1$ for all $x \in X$.

Proof: Let $\beta(X)$ be the Stone-Cech compactification of X . Regard X as being a subspace of $\beta(X)$. Then K is compact in $\beta(X)$ and hence closed. For i , $1 \leq i \leq n$, there exists an open set O_i such that $V_i = O_i \cap X$. Then $\{\beta(X) \sim K, O_1, \dots, O_n\}$ is an open cover for $\beta(X)$. So by prop. 3 of §4, of [7], for each i , $0 \leq i \leq n$, there exists a $\psi_i: \beta(X) \rightarrow [0,1]$, continuous, such that $\sum_{i=0}^n \psi_i(x)$

equals 1 for all $x \in \beta(X)$ and such that $\text{Supp } \psi_0 \subset \beta(X) \sim K$ and $\text{Supp } \psi_i \subset O_i$ for all $i, 1 \leq i \leq n$. Define $\phi_i = \psi_i|_X$ for all $i, 0 \leq i \leq n$. Notice that for $1 \leq i \leq n$, $\text{Supp } \phi_i \subset O_i \cap X = V_i$ and $\text{Supp } \phi_0 \subset (\beta(X) \sim K) \cap X = X \sim K$. QED

13.2 Remark. A version of this remains true if X is only assumed to be completely regular, since the map from X to $\beta(X)$ is still continuous and open onto its range.

13.3 Lemma. Let E be a locally convex Hausdorff space. Let X be a topological space. Then for all $\mu \in c_c(X, E)'$ there exists a compact set $K \subset X$ such that if $f \in c(X, E)$ and $f(k) = 0$ for all $k \in K$, then $\mu(f) = 0$.

Proof: Suppose there exists a $\mu \in c_c(X, E)'$ for which this is not true. Then for all K compact in X , there exists an $r: X \rightarrow E$ such that $r(k) = 0$ for all $k \in K$ and $\mu(r) \neq 0$. Choose a scalar α such that $\mu(\alpha r) = 1$. Define $s_K = \alpha r$. Let D be the directed set of all compact subsets of X ordered by inclusion. Then the s_K form a net defined on D such that $s_K \rightarrow 0$ in $c_c(X, E)$; since if K is compact and U is a neighborhood of zero then $A \supseteq K$ implies for all $x \in K$, $s_A(x) = 0 \in U$.

So $\mu(s_K) \rightarrow \mu(0) = 0$ because μ is continuous. But $\mu(s_K) = 1$ for all K . So $\mu(s_K) \rightarrow 1$. Since the scalars are Hausdorff, these two statements imply that

1 = 0. Contradiction.

QED

13.4 Definition. If f is a vector-valued continuous function, then the support of f , denoted $\text{Supp } f$, is defined to be $\{x : f(x) \neq 0\}^-$.

13.5 Theorem. Let E be a locally convex Hausdorff space and X be a Tychonoff space. Then

- 1) for all $\mu \in c_c(X, E)'$, there exists a smallest closed set $A \subset X$ with the property that if $f: X \rightarrow E$ is continuous and $\text{Supp } f \subset X \sim A$, then $\mu(f) = 0$; and
- 2) this smallest closed set is compact.

Proof: Let $\mu \in c_c(X, E)'$. Let \mathcal{F} be the set of all closed sets $C \subset X$ which have the property that if $f \in c(X, E)$ with $\text{Supp } f \subset X \sim C$, then $\mu(f) = 0$. Let $Q = \bigcap \mathcal{F}$. Q is closed and if I can show that Q has the above property, then it will certainly be the smallest such set. Note that Q is compact, since \mathcal{F} has a compact member by the last lemma.

Suppose $f: X \rightarrow \mathbb{K}$ is continuous and $\text{Supp } f$ is contained in $X \sim Q$. Let F_0 be a compact element of \mathcal{F} . Then $F_0 \subset X \subset (X \sim \text{Supp } f) \cup \left[\bigcup_{F \in \mathcal{F}} (X \sim F) \right]$. Since F_0 is compact, there exists a finite subcollection $\{F_1, \dots, F_n\} \subset \mathcal{F}$ such that $F_0 \subset (X \sim \text{Supp } f) \cup \bigcup_{i=1}^n (X \sim F_i)$. So there exists $\{\phi_0, \dots, \phi_{n+1}\} \subset c(X)$ such that $\text{Supp } \phi_i \subset X \sim F_i$ for $0 \leq i \leq n$, $\text{Supp } \phi_{n+1} \subset X \sim \text{Supp } f$, and $\sum_{i=0}^{n+1} \phi_i(x) = 1$ for all $x \in X$. So

$\mu(f) = \mu((\sum_{i=0}^{n+1} \phi_i)f) = \sum_{i=0}^{n+1} \mu(\phi_i f)$. Now for $0 \leq i \leq n$,
 $\text{Supp } \phi_i f \subset \text{Supp } \phi_i \subset X \sim F_i$. So for $0 \leq i \leq n$, $\mu(\phi_i f) = 0$,
 because $\{F_i : 0 \leq i \leq n\} \subset \mathcal{F}$. Hence $\mu(f) = \mu(\phi_{n+1} f)$. But
 $\phi_{n+1} f = 0$, since $\text{Supp } f \cap \text{Supp } \phi_{n+1} = \emptyset$. So $\mu(\phi_{n+1} f)$
 equals 0. Hence $\mu(f) = 0$. QED

13.6 Definition. If X is a Tychonoff space, E is locally convex Hausdorff and $\mu \in c_c(X, E)'$, then call the set which is described in the statement of 13.5 the support of μ and denote it by $\text{Supp } \mu$.

13.7 Theorem. Let X be a Tychonoff space and E be locally convex Hausdorff. If $\mu \in c_c(X, E)'$ and $f \in c(X, E)$ such that $f(x) = 0$ for all $x \in \text{Supp } \mu$, then $\mu(f) = 0$.

Proof: Let $A = \text{Supp } \mu$. Suppose $f \in c(X, E)$ and $f(x) = 0$ for all $x \in A$. Let U be a balanced, open, neighborhood of zero and let $K \subset X$ be compact. Choose an open neighborhood V of zero such that $0 \in V \subset V^- \subset U$. Let $K' = f^{-1}(E \sim U) \cap K$. K' is compact, since $E \sim U$ is closed. Let $B = f^{-1}(V^-)$. B is closed and $K' \cap B = \emptyset$. Since X is completely regular, there exists a continuous function $h: X \rightarrow [0, 1]$ such that $h(K') = \{1\}$ and $h(B) = \{0\}$. Now $\text{Supp } hf \subset \text{Supp } h \subset f^{-1}(E \sim V) \subset X \sim A$. So $\mu(hf) = 0$, i.e. $hf \in \mu^{-1}(\{0\})$. Also $(hf)(x) - f(x) \in U$, for all $x \in K$, because 1) if $x \in K'$ then $(hf)(x) = h(x)f(x) = f(x)$, and 2) if $x \in K \sim K'$ then $f(x) \in U$ and $f(x) - h(x)f(x) = (1 - h(x))f(x) \in U$ since

$0 \leq 1 - h(x) \leq 1$ and U is balanced. So in the topology of $c_c(X, E)$, f adheres to the closed set $\mu^{-1}(\{0\})$.

So $f \in \mu^{-1}(\{0\})$, i.e. $\mu(f) = 0$. QED

13.8 Lemma. If X is a Tychonoff space, E is locally convex Hausdorff, and $\mu \in c_c(X, E)'$, then $\mu = 0$ iff $\text{Supp } \mu = \emptyset$.

Proof: Trivial.

13.9 Lemma. Suppose X is a Tychonoff space and E is locally convex and Hausdorff. Then

1) if $c_c(X, E)'$ is given the $\sigma(c_c(X, E)', c_c(X, E))$ topology and $\{\mu_\alpha\}_{\alpha \in I}$ is a net in $c_c(X, E)'$ converging to μ , then $\text{Supp } \mu \subset [\cup\{\text{Supp } \mu_\alpha : \alpha \in J\}]^-$, where J is any cofinal subset of I ; and

2) if λ is a non-zero scalar and $\mu, \nu \in c_c(X, E)'$ then $\text{Supp } \lambda\mu = \text{Supp } \mu$ and $\text{Supp } (\mu + \nu) \subset \text{Supp } \mu \cup \text{Supp } \nu$.

Proof: 1) Without loss of generality we may assume that $J = I$. Suppose $f \in c(X, E)$ and $\text{Supp } f \subset X \sim [\cup\{\text{Supp } \mu_\alpha : \alpha \in I\}]^- \subset \cap\{(X \sim \text{Supp } \mu_\alpha) : \alpha \in I\}$. Then $\mu_\alpha(f) = 0$ for all $\alpha \in I$. But $\mu_\alpha(f) \rightarrow \mu(f)$. So $\mu(f) = 0$. Hence $\text{Supp } \mu \subset [\cup\{\text{Supp } \mu_\alpha : \alpha \in I\}]^-$.

2) Suppose $f \in c(X, E)$ and $\text{Supp } f \subset X \sim (\text{Supp } \mu \cup \text{Supp } \nu) = (X \sim \text{Supp } \mu) \cap (X \sim \text{Supp } \nu)$. Then $\mu(f)$ and $\nu(f)$ both equal zero. So $(\mu + \nu)(f) = 0$. $\text{Supp } \mu \cup \text{Supp } \nu$ is closed, so $\text{Supp } (\mu + \nu) \subset \text{Supp } \mu \cup \text{Supp } \nu$.

Suppose $f \in c(X, E)$ and $\text{Supp } f \subset X \sim \text{Supp } \mu$.

Then $\mu(f) = 0$. So $\lambda\mu(f) = 0$. Hence $\text{Supp } \lambda\mu \subset \text{Supp } \mu$.
 Since $\lambda \neq 0$, the same statement can be made of $1/\lambda$.
 Thus $\text{Supp } \mu = \text{Supp } (1/\lambda)(\lambda\mu) \subset \text{Supp } \lambda\mu$. So $\text{Supp } \lambda\mu =$
 $\text{Supp } \mu$. QED

13.10 Corollary. Suppose X is a Tychonoff space and E is locally convex Hausdorff. Let $A \subset X$. Then $\{\mu \in c_c(X, E)' : \text{Supp } \mu \subset A\}$ is a linear subspace of $c_c(X, E)'$, and if A is closed in X , then it is a $\sigma(c_c(X, E)', c_c(X, E))$ -closed subspace.

Proof: The proof is a straight forward application of 13.8 and 13.9. QED

13.11 Theorem. Let X be a Tychonoff space and A be a closed subset of X . Suppose either A is compact or X is hemi-compact. Let E be a Frechet space. Let $a: c_c(X, E) \rightarrow c_c(A, E)$ be the restriction map. Give both $c_c(A, E)'$ and $c_c(X, E)'$ either the topology of pointwise convergence or the topology of compact convergence. Let $Q = \{\mu \in c_c(X, E)' : \text{Supp } \mu \subset A\}$. Give Q the relative topology from $c_c(X, E)'$. Then for all $\mu \in c_c(A, E)'$, $t_a(\mu) = \mu \circ a \in Q$. Moreover $t_a: c_c(A, E)' \rightarrow Q$ is an isomorphism in the category \mathcal{C} .

Proof: Let $\sigma: c_c(A, E) \rightarrow c_c(X, E)$ be a continuous map such that $a \circ \sigma = 1_{c_c(A, E)}$ (cf. 11.2). Define a map $\Phi: Q \rightarrow c(c_c(A, E))$ by $\Phi(\mu) = \mu \circ \sigma$.

Claim: For all $\mu \in Q$, $\Phi(\mu)$ is linear, i.e. Φ maps Q into $c_c(A, E)'$.

Proof of claim: Suppose $\mu \in Q$. Let $f, g \in c(A, E)$ and $\delta, \lambda \in K$. Then since a is linear we have $a[\sigma(\delta f + \lambda g)] = \delta f + \lambda g = a[\delta\sigma(f) + \lambda\sigma(g)]$. So since $\text{Supp } \mu \subset A$, we have by 13.7 that $\mu(\sigma(\delta f + \lambda g)) = \mu(\delta\sigma(f) + \lambda\sigma(g)) = \delta\mu(\sigma(f)) + \lambda\mu(\sigma(g))$. Hence $[\Phi(\mu)](\delta f + \lambda g) = \delta\Phi(\mu)(f) + \lambda\Phi(\mu)(g)$. Hence $\Phi(\mu)$ is linear. QED on claim

Note that $\Phi: Q \rightarrow c_c(A, E)'$ is continuous and linear.

Claim: For all $\mu \in c_c(A, E)'$, ${}^t a(\mu) = \mu \circ a \in Q$.

Proof of claim: Suppose $f \in c(X, E)$ and $\text{Supp } f \subset X \sim A$. Then $[{}^t a(\mu)](f) = \mu(a(f)) = 0$, since $a(f) = 0$. So $\text{Supp } {}^t a(\mu) \subset A$ and hence ${}^t a(\mu) \in Q$. QED on claim

Claim: $\Phi \circ {}^t a = 1_{c_c(A, E)'}$.

Proof of claim: Let $\mu \in c(A, E)'$. Then $\Phi({}^t a(\mu)) = [{}^t a(\mu)] \circ \sigma = \mu \circ a \circ \sigma = \mu$. QED on claim

Claim: ${}^t a \circ \Phi = 1_Q$.

Proof of claim: Let $\mu \in Q$. Then ${}^t a(\Phi(\mu)) = [\Phi(\mu)] \circ a = \mu \circ \sigma \circ a$. Let $f \in c(X, E)$. Then $a(f) = (a \circ \sigma)(a(f)) = a((\sigma \circ a)(f))$. So since $\text{Supp } \mu$ is contained in A , we have $\mu(f) = \mu((\sigma \circ a)(f))$. So $({}^t a \circ \Phi)(\mu)(f) = \mu(f)$. And thus $({}^t a \circ \Phi)(\mu) = \mu$, for all $\mu \in Q$. QED on claim

Hence ${}^t a: c_c(A, E)' \rightarrow Q$ is an isomorphism with inverse Φ . QED

The idea for the following comes from a part of a proof given by Shirota [43]. Also see Nachbin [34].

13.12 Theorem. Let E be a locally convex Hausdorff space. Let X be a Tychonoff space. If $A \subset c_c(X, E)'$ is $\sigma(c_c(X, E)', c_c(X, E))$ -bounded, then $\{\cup\{\text{Supp } \mu : \mu \in A\}\}^-$ is topologically precompact in X .

Proof: Let $C = \cup\{\text{Supp } \mu : \mu \in A\}$. Suppose C is not topologically precompact in X . Then there exists a discrete collection \mathcal{C} of open sets such that $\{V \in \mathcal{C} : C \cap V \neq \emptyset\}$ is infinite.

Let $N =$ the natural numbers. Choose a countably infinite subcollection $\{U_i\}_{i \in N}$ of \mathcal{C} such that $i \neq j$ implies $U_i \neq U_j$ and such that $C \cap U_i \neq \emptyset$ for all $i \in N$. For each $i \in N$, choose a $\mu_i \in A$ such that $\text{Supp } \mu_i \cap U_i \neq \emptyset$.

Claim: There exists a sequence $\{f_i\}_{i \in N} \subset c(X, E)$ and a strictly monotone function $r: N \rightarrow N$ such that for all $i \in N$

- 1) $\mu_{r(i)}(f_1 + \dots + f_i) = i$,
 - 2) $U_{r(i)} \subset X \sim \cup\{\text{Supp } \mu_{r(j)} : j \in N \text{ and } j < i\}$,
- and 3) $\text{Supp } f_i \subset U_{r(i)}$.

Proof of claim: Assume $i \in N$. Suppose $\{f_j : j < i\}$ and $r\{j : 1 \leq j < i\}$ have already been defined.

Define $r(0) = 0$. Now $\cup_{j=1}^{i-1} \text{Supp } \mu_{r(j)}$ is compact by 13.6. Hence it is topologically precompact in X . So $\{p \in N : U_p \cap \cup_{j=1}^{i-1} \text{Supp } \mu_{r(j)} \neq \emptyset\}$ is finite. Let $\gamma = \sup(\{p \in N : U_p \cap \cup_{j=1}^{i-1} \text{Supp } \mu_{r(j)} \neq \emptyset\} \cup \{0\})$. Define $r(i) = \sup\{r(i-1) + 1, \gamma + 1\}$. Note that $r(i) > r(i-1)$ and $U_{r(i)} \subset X \sim \cup\{\text{Supp } \mu_{r(j)} : 1 \leq j < i\}$.

Now there exists a $g: X \rightarrow E$ such that $\mu_{r(i)}(g) \neq 0$ and $\text{Supp } g \subset U_{r(i)}$, because otherwise we would have $\text{Supp } \mu_{r(i)} \subset X \sim U_{r(i)}$ but it is not because $\text{Supp } \mu_{r(i)} \cap U_{r(i)} \neq \emptyset$. Define

$$f_i = \left[\frac{i - \mu_{r(i)}(\sum_{j=1}^{i-1} f_j)}{\mu_{r(i)}(g)} \right] g. \quad \text{Then } \mu_{r(i)}(\sum_{j=1}^i f_j) = i.$$

Also $\text{Supp } f_i \subset \text{Supp } g \subset U_{r(i)}$. So conditions 1), 2), and 3) are satisfied. Thus by induction we obtain a sequence f_j and a function r satisfying the three conditions. QED on claim

Claim: $\sum_{j \in \mathbb{N}} f_j \in c(X, E)$.

Proof of claim: Let $x \in X$. Since $\{U_{r(j)} : j \in \mathbb{N}\}$ is a discrete collection, there exists a neighborhood V of x which intersects at most one of the $U_{r(j)}$. If V intersects at most $U_{r(i)}$, then $[\sum_{j \in \mathbb{N}} f_j]|_V = f_i|_V$ due to 3) of the last claim. So $\sum_{j \in \mathbb{N}} f_j$ is continuous at x . QED on claim

Let $i \in \mathbb{N}$. What is $\mu_{r(i)}(\sum_{j \in \mathbb{N}} f_j) = \mu_{r(i)}(\sum_{j=1}^i f_j) + \mu_{r(i)}(\sum_{j \geq i+1} f_j)$? Well $[\sum_{j \geq i+1} f_j](x)$ equals zero for all $x \in \text{Supp } \mu_{r(i)}$, because $f_j(x) = 0$ for all $x \in \text{Supp } \mu_{r(i)}$ if $j \geq i+1$; since if $j \geq i+1$, $\text{Supp } f_j \subset U_{r(j)} \subset X \sim \bigcup_{p=1}^{j-1} \text{Supp } \mu_{r(p)} \subset X \sim \text{Supp } \mu_{r(i)}$. So $\mu_{r(i)}(\sum_{j \geq i+1} f_j) = 0$ by 13.7. Hence $\mu_{r(i)}(\sum_{j \in \mathbb{N}} f_j) = \mu_{r(i)}(\sum_{j=1}^i f_j) = i$. Hence $\{\mu_{r(i)}\}_{i \in \mathbb{N}}$ is not weakly bounded. Hence A is not either, since $\{\mu_{r(i)}\}_{i \in \mathbb{N}} \subset A$. Contradiction. Hence $\bigcup \{\text{Supp } \mu : \mu \in A\}$ is topologically precompact in X . But then by 12.2

$[\cup\{\text{Supp } \mu : \mu \in A\}]^-$ is topologically precompact in X .

QED

13.13 Corollary. If X is a topologically p -complete space, E is any locally convex Hausdorff space, and $B \subset c_c(X, E)'$ is $\sigma(c_c(X, E)', c_c(X, E))$ -bounded, then there exists a compact set K in X such that for all $\mu \in B$, $\text{Supp } \mu \subset K$.

Proof: Let $K = [\cup\{\text{Supp } \mu : \mu \in B\}]^-$. K is closed and topologically precompact in X and hence is compact. QED

The following was proved by Shirota [43] and Nachbin [34] for the special case of scalar-valued functions. This proof uses their ideas.

13.14 Corollary. If X is a topologically p -complete space and E is a Frechet space, then $c_c(X, E)$ is barrelled.

Proof: Let B be a $\sigma(c_c(X, E)', c_c(X, E))$ -bounded set of $c_c(X, E)'$. Using the last corollary, let K be a compact subset of X such that $\text{Supp } \mu \subset K$ for all $\mu \in B$.

Let $a: c_c(X, E) \longrightarrow c_c(K, E)$ be the restriction map. By 13.11, ${}^t a: c_c(K, E)' \longrightarrow Q$ is a linear topological isomorphism provided that $c_c(K, E)'$ and $c_c(X, E)'$ are given the topology of pointwise convergence and Q is defined to be equal to $\{\mu \in c_c(X, E)' : \text{Supp } \mu \subset K\}$ and is given the relative topology from $c_c(X, E)'$. But $c_c(K, E)$ is a Frechet space and is hence barrelled. So

$(t_a)^{-1}(B)$ is equicontinuous. So
 $B = t_a[(t_a)^{-1}(B)] = [(t_a)^{-1}(B)] \circ a$ is equicontinuous,
 since a is continuous. QED

Shirota [43] and Nachbin [34] proved the following theorem for the special case of scalar-valued functions.

13.15 Theorem. Let X be a Tychonoff space and E be a non-zero locally convex Hausdorff space. If $c_c(X, E)$ is barrelled, then X is topologically p -complete.

Proof: Suppose P is closed and topologically precompact in X . Further suppose that P is not compact. Let $e \in E \sim \{0\}$. Since E is Hausdorff, there exists a closed, balanced, convex neighborhood U of zero in E such that $e \notin U$. Let $Q = \{f \in c(X, E) : f(x) \in U \forall x \in P\}$. Q is a closed, convex, balanced subset of $c_c(X, E)$.

Claim: Q is not a neighborhood of zero.

Proof of claim: Assume there exists a compact set $K \subset X$ and a neighborhood V of zero in E such that $\{f \in c(X, E) : f(k) \in V \text{ for all } k \in K\} \subset Q$. Then $P \not\subset K$, because if it were P would be compact. So there exists a $p \in P$ such that $p \notin K$. Since X is completely regular, there exists a $f: X \rightarrow [0, 1]$ such that $f(p) = 1$ and $f(k) = 0$ for all $k \in K$. Consider the function $g: X \rightarrow E$ such that $g(x) = f(x)e$. Now $g \in Q$, since $g(x) = 0 \in V$ for all $x \in K$. Thus $e = g(p) \in U$. But $e \notin U$. Contradiction. QED on claim

Hence since $c_c(X, E)$ is assumed barrelled, Q can-

not be absorbing. So there exists a $f_0 \in c(X, E)$ such that for all $\lambda > 0$, $\lambda f_0 \notin Q$.

Since P is topologically precompact in X , by 12.33 we

have that $f_0(P)$ is precompact and hence bounded in E . So there exists a $\lambda > 0$ such that $\lambda f_0(P) \subset U$.

So $\lambda f_0 \in Q$. Contradiction. So P must be compact.

Thus X is topologically p -complete. QED

Frankly I would like necessary and sufficient conditions for $c_c(X, E)$ to be p -determined, but I have had no success in this direction.

Section 14 - Consequences of $c_c(X)$ being barrelled.

In this section, I apply the results of the last section to get results about $M(X)$, where X is a topologically p -complete k -space.

14.1 Lemma. If X is a topologically p -complete k -space, then $M(X) = [c_c(X)]^P$.

Proof: $c_c(X)$ is barrelled by 13.14. Hence it is p -determined. So the coreflection of $c_c(X)$ in the category of p -determined spaces is itself, i.e. $C(X) = c_c(X)$.

Hence the result follows from the definition of $M(X)$ in 10.5. QED

14.2 Theorem. Let X be a topologically p -complete k -space. Then T is a bounded subset of $M(X)$ iff there exists a compact subset K of X and a number $L > 0$ such that

$$T \subset L \cdot [\text{the closed, convex, balanced hull of } \varepsilon(K)].$$

Proof: (\Rightarrow) Recall that $\varepsilon: X \rightarrow M(X)$ is the evaluation map. By 14.1, $M(X) = [c_c(X)]^P$. If T is bounded in $M(X)$, then it is $\sigma(c_c(X)', c_c(X))$ -bounded. Thus T is equicontinuous, since $c_c(X)$ is barrelled. Hence T° is a neighborhood of zero in $c_c(X)$. So there exists an $L > 0$ and a $K \subset X$ compact such that

$$\{f \in c_c(X) : |f(k)| \leq L^{-1} \text{ for all } k \in K\} \subset T^\circ.$$

Suppose $f \in [\varepsilon(K)]^\circ$. Then for all $k \in K$
 $1 \geq |\varepsilon(k)(f)| = |f(k)|$. Thus $|L^{-1}f(k)| \leq L^{-1}$ for all
 $k \in K$. So $L^{-1}f \in T^\circ$, i.e. $f \in LT^\circ$. So $[\varepsilon(K)]^\circ \subset LT^\circ$.
Hence $[\varepsilon(K)]^{\circ\circ} \supset (LT^\circ)^\circ = L^{-1}T^{\circ\circ} \supset L^{-1}T$. But $[\varepsilon(K)]^{\circ\circ}$
is the closed convex balanced hull of $\varepsilon(K)$. QED on \Rightarrow
 (\Leftarrow) If K is compact, then $\varepsilon(K)$ is compact.
So $L[\varepsilon(K)]^{\circ\circ}$ is precompact. QED

14.3 Theorem. Let X be a topologically p -complete
 k -space and let E be a p -complete locally convex space.
Let $\varepsilon: X \rightarrow M(X)$ be the canonical map. Then the map
 $j: \text{hom}_p(M(X), E) \rightarrow c_c(X, E)$ defined by $j(T) = T \circ \varepsilon$ is
an isomorphism in the category \mathcal{C} .

Proof: The proof of theorem 10.6 proves that j is an
isomorphism in the category of vector spaces. Thus in
order to show that j is a \mathcal{C} -isomorphism we only need
show that it is bicontinuous.

The continuity on j follows immediately from the
fact that ε is continuous and thus sends compact sets
to compact sets.

Suppose $R \subset M(X)$ is precompact and $U \subset E$ is a
closed, balanced, convex neighborhood of zero. By 14.2,
there exists a compact subset K of X and a number L
greater than zero such that

$R \subset L \cdot [\text{the closed, convex, balanced hull of } \varepsilon(K)]$.

Claim: $j(\{T : T(R) \subset U\}) \supset \{f : f(K) \subset L^{-1}U\}$.

Proof of claim: Let $f \in c(X, E)$ such that $f(K) \subset$
 $L^{-1}U$. Let $\tilde{f}: M(X) \rightarrow E$ be the unique map such that

$\bar{f} \circ \varepsilon = f$. So $\bar{f}(\varepsilon(K)) \subset L^{-1}U$, i.e. $\varepsilon(K) \subset \bar{f}^{-1}(L^{-1}U)$.
 Now $\bar{f}^{-1}(L^{-1}U)$ is closed, convex, and balanced; since $L^{-1}U$ is. So the closed convex balanced hull of $\varepsilon(K)$ is contained in $\bar{f}^{-1}(L^{-1}U)$. So $R \subset \bar{f}^{-1}(U)$. Hence $\bar{f}(R) \subset U$ and $j(\bar{f}) = \bar{f} \circ \varepsilon = f$. QED on claim

But the claim proves that j^{-1} is continuous.

Hence j is an isomorphism in the category \mathcal{C} . QED

14.4 Corollary. Let X be a topologically p -complete k -space. Let F be a Frechet space. Then $\text{hom}_p(M(X), F)$ is barrelled.

Proof: By the last theorem, $c_c(X, F) \cong \text{hom}_p(M(X), F)$.

But $c_c(X, F)$ is barrelled by 13.14. QED

The following theorem leads to definition 14.6.

14.5 Theorem. Suppose E is a locally convex space, then the following two properties are equivalent:

1) If $A \subset E$ is convex and balanced such that for all $C \subset E$ compact which contain zero, $A \cap C$ is a neighborhood of zero for the relative topology on C induced by E , then A is a neighborhood of zero in E .

2) If F is a locally convex space and $T: E \rightarrow F$ is a linear map such that $T|_C: C \rightarrow F$ is continuous at zero, for all $C \subset E$ which are compact and contain zero, then T is continuous.

Proof: 1) \Rightarrow 2). Suppose T is as stated, let U be a

balanced, convex neighborhood of zero in F . Then $T^{-1}(U)$ is balanced and convex. Suppose C is a compact set containing zero. Then $T|_C$ is continuous at 0 . So $(T|_C)^{-1}(U) = C \cap T^{-1}(U)$ is a neighborhood of zero in the relative topology on C . So $T^{-1}(U)$ is a neighborhood of zero in E since we are assuming 1). Hence T is continuous at zero and therefore continuous.

2) \Rightarrow 1). Let \mathcal{O} be the set of those subsets A of E such that A is convex, balanced, and $A \cap C$ is a neighborhood of zero in C for all C which are compact in E and contain zero.

Claim: \mathcal{O} is a neighborhood base at zero for a locally convex topology on the vector space E .

Proof of claim: Each element of \mathcal{O} is absorbing by 1.1 and 1.2. Thus $A \in \mathcal{O}$ implies $A \neq \emptyset$. Suppose $A_1, A_2 \in \mathcal{O}$. Then $A_1 \cap A_2$ is convex and balanced. Let C be compact with $0 \in C$. Then $A_1 \cap C$ and $A_2 \cap C$ are both neighborhoods of zero in C , and hence so is $(A_1 \cap A_2) \cap C$. Thus $A_1 \cap A_2 \in \mathcal{O}$. Hence \mathcal{O} is a filter base.

Let $\lambda > 0$ and $A \in \mathcal{O}$. λA is convex and balanced. Suppose C is compact and $0 \in C$. Then $\lambda^{-1}C$ is also compact and contains zero. So there exists a neighborhood V of zero such that $V \cap \lambda^{-1}C \subset A \cap \lambda^{-1}C$. But then $(\lambda V) \cap C \subset (\lambda A) \cap C$. So $(\lambda A) \cap C$ is a neighborhood of zero in C , because λV is a neighborhood of zero. So $\lambda A \in \mathcal{O}$.

Hence by Proposition 2.4.5 of [19], \mathcal{O} is a neigh-

neighborhood base at zero for a locally convex topology on E . Call E with this topology E_f . Let $I: E \rightarrow E_f$ be the identity map. I is linear. Suppose C is compact in E and $0 \in C$. Then $I|_C$ is continuous at zero into E_f ; because if $A \in \mathcal{O}_f$, which is a base at zero for E_f , then $(I|_C)^{-1}(A) = A \cap C$ which is a neighborhood of zero in C . So because we are assuming 2), I is continuous.

Hence if $A \in \mathcal{O}$, $I^{-1}(A) = A$ is a neighborhood of zero in E . Thus property 1) holds. QED

14.6 Definition. A locally convex space satisfying either of the equivalent conditions of theorem 14.5 will be called an ℓk -space, [i.e. linear k -space].

This concept will be used in conjunction with the functor M .

14.7 Lemma. Let E be a locally convex Hausdorff space. If E is complete, then $E^{\mathcal{P}}$ is an ℓk -space.

Proof: E complete implies that $E^{\mathcal{P}} = (E', \mu(E', E))$ where $\mu(E', E)$ is the finest locally convex topology on E' which induces on equicontinuous subsets the same topology as the topology of pointwise convergence (cf. §21.9, (8) of [27] and the first sentence of p. 252 of [19]).

Let \mathcal{M} be the set of all subsets A of E' such that A is absorbing, balanced, convex, and $A \cap M$ is a neighborhood of zero in M with the relative topology induced by $\sigma(E', E)$, for all equicontinuous sets M

containing zero. Now \mathcal{M} is a base at zero for $\mu(E', E)$ (cf. proposition 2 of p.251 of [19]).

Suppose $A \subset E'$ is balanced, convex and $A \cap C$ is a neighborhood of zero in C with the relative topology from E^P , for all $C \subset E'$ which contain zero and which are compact in E^P . Note by 1.1 and 1.2 that A is also absorbing. Suppose M is equicontinuous and contains zero. Let $M^- =$ the $\sigma(E', E)$ -closure of M . Then M^- is $\sigma(E', E)$ -compact (by Alaoglu) and hence M^- is compact in E^P , since M^- is equicontinuous (cf. Propositions 4 and 5 of Chap. 3, §3, of [4]). M^- also contains zero. Hence there exists a neighborhood of zero V in E^P such that $V \cap M^- \subset A \cap M^-$. But on equicontinuous sets, the topologies of precompact and pointwise convergence agree. So there exists a $\sigma(E', E)$ -neighborhood of zero U such that $M^- \cap U \subset M^- \cap V \subset M^- \cap A$. So $M \cap U \subset M \cap A$. Hence $M \cap A$ is a $\sigma(E', E)$ neighborhood of zero in M . So $A \in \mathcal{M}$. So A is a neighborhood of zero in E^P . Hence the first of the two equivalent conditions which define ℓk -spaces is satisfied. So E^P is an ℓk -space. QED

The following result will be needed in the next theorem.

14.8 Lemma. If E is a barrelled, locally convex Hausdorff topological vector space, then E^P is a semi-Montel space.

Proof: Let A be a closed, bounded subset of $E^{\mathbb{P}}$. Then A is $\sigma(E', E)$ bounded, hence equicontinuous because E is barrelled. Let A^- = the $\sigma(E', E)$ -closure of A . A^- is $\sigma(E', E)$ -compact and equicontinuous. Hence A^- is compact in $E^{\mathbb{P}}$ (cf. prop. 5 of chap. 3, §3, of [4]). But A is closed in $E^{\mathbb{P}}$ and $A \subset A^-$. Hence A is compact as a subspace of $E^{\mathbb{P}}$. QED

14.9 Theorem. Let X be a topologically p -complete k -space. Then

- 1) $M(X)$ is semi-Montel space and a ℓk -space.
- 2) The following are equivalent:
 - a) X is discrete
 - b) $M(X)$ is a Montel space
 - c) $M(X)$ is barrelled
 - d) $M(X)$ is quasibarrelled
 - e) $M(X)$ is reflexive
 - f) $M(X) = c_c(X)'$ with the topology of bounded convergence.
- 3) The following are equivalent:
 - a) $M(X)$ is metrizable
 - b) X is finite.
- 4) $M(X)$ is a dF space iff X is hemi-compact, and $M(X)$ is a dB space iff X is compact.
- 5) If $X = \bigoplus_{\alpha \in I} X_{\alpha}$ where each X_{α} is a hemi-compact, k -space; then $M(X)$ is complete.
- 6) If X is locally compact and paracompact, then $M(X)$ is complete.

7) If X has a topology which is at least as fine as another topology \mathcal{J} on X such that (X, \mathcal{J}) is a separable metric space, then bounded subsets of $M(X)$ are metrizable.

Proof: 1) $c_c(X)$ is complete since X is a k -space. So by 14.1 and 14.7, $M(X)$ is an ℓk -space. $c_c(X)$ is barrelled by 13.14, so by 14.1 and 14.8, $M(X)$ is a semi-Montel space. QED on 1)

2) If X is discrete, then $c_c(X)$ is Montel by theorem 10 of [48]. Thus $M(X) = [c_c(X)]^P$ = the strong dual of $c_c(X)$. By prop. 9 on p. 236 of [19], the strong dual of a Montel space is Montel, hence $M(X)$ is Montel. Now Montel \Rightarrow reflexive \Rightarrow barrelled \Rightarrow quasibarrelled. By 1), $M(X)$ is semi-Montel and so quasicomplete. So $M(X)$ quasibarrelled $\Rightarrow M(X)$ is barrelled, since quasicomplete and quasibarrelled \Rightarrow barrelled. So we have shown $a \Rightarrow b \Rightarrow e \Rightarrow c \Rightarrow d \Rightarrow c$. But $M(X)$ is semi-Montel. So $c \Rightarrow b$.

Claim: The strong dual of $M(X) = c_c(X)$.

Proof of claim: Now $[M(X)]^P$ = the strong dual of $M(X)$, since $M(X)$ is semi-Montel. Also $c_c(X) = [c_c(X)]^{PP} = [M(X)]^P$ since $c_c(X)$ is p -reflexive. So the strong dual of $M(X) = c_c(X)$. QED on claim

Thus if $M(X)$ is a Montel space, then $c_c(X)$ is Montel because the strong dual of a Montel space is a Montel space. But then X is discrete (again by theorem 10 of [48]). So we have shown the equivalence of every-

thing except f).

Suppose $M(X)$ is reflexive. Then $M(X)$ equals the strong dual of the strong dual of $M(X)$. But by the claim above, the strong dual of $M(X)$ equals $c_c(X)$. Thus $e \Rightarrow f$.

Suppose f) is satisfied. Then again by use of the claim above, the strong dual of the strong dual of $M(X)$ equals $M(X)$. Thus $M(X)$ is reflexive. So $f \Rightarrow e$.

QED on 2)

3) Suppose $M(X)$ is metrizable. Then $M(X)$ is bornological and hence quasibarrelled (cf. Prop. 3, § 3.7 of [19]). So by 2), X is discrete.

$M(X)$ is a Frechet space by 6.3. Thus by 6.3, $c_c(X) \cong [c_c(X)]^{PP} = [M(X)]^P$ is a dF space. So $c_c(X)$ is hemi-compact. Let $\{A_n : n \in \omega\}$ be a collection of compact sets of $c_c(X)$ such that $\bigcup_{n \in \omega} A_n = c_c(X)$.

Since each A_n is compact, for all $x \in X$ $\{a(x) : a \in A_n\}$ is precompact in the scalars. So define a function $p: \omega \times X \rightarrow \mathbb{R}^+$ by $p(n, x) = \sup\{|a(x)| : a \in A_n\}$. Suppose X is infinite. Then let $q: \omega \rightarrow X$ be any injective function. Define a function $f: X \rightarrow \mathbb{K}$ as follows

$$f(x) = \left\{ \begin{array}{ll} 0 & \text{if } x \notin \text{Range } q \\ p(n, x) + 1 & \text{if } x = q(n) \end{array} \right\}.$$

f is continuous since X is discrete. And for all $j \in \omega$, $f \notin A_j$; because $f \in A_j$ implies that $|f(q(j))| \leq p(j, q(j))$. But $f(q(j)) = p(j, q(j)) + 1$.

But recalling how the A_j were defined we see that we have a contradiction. Thus X must be finite.

For the converse, if X is finite, then $M(X) = \mathbb{K}^n$ for some n . Hence $M(X)$ is metrizable. QED on 3).

4) X is hemi-compact iff $c_c(X)$ is Frechet iff $M(X)$ is a dF space (cf. 6.3, and theorems 7 and 8 of [2]).

Suppose $c_c(X)$ is a Banach space. Let B be a bounded neighborhood of zero in $c_c(X)$. Then there exists a compact set K and an $\varepsilon > 0$ such that

$\{f : |f(k)| < \varepsilon \text{ for all } k \in K\} \subset B$. Suppose $K \neq X$.

Then there exists a $x_0 \in X \setminus K$. By complete regularity of X , for all $n \in \omega$ there exists a $f_n \in c(X)$ such that $f_n(x_0) = n$ and $f_n(K) \subset \{0\}$. So for all $n \in \omega$, $f_n \in B$. But then B is not bounded. Contradiction.

Hence $K = X$. So X is compact.

Thus X is compact iff $c_c(X)$ is Banach iff $M(X)$ is dB (cf. 6.7). QED on 4)

5) The functor M has a coadjoint. Hence M preserves colimits. Since coproducts in the category of topological spaces and the full subcategory of k -spaces agree (cf. C6), $M(\oplus X_\alpha) = \oplus M(X_\alpha)$. Where the right hand coproduct is in the category of p -complete spaces. But coproducts in that category and in the category of locally convex Hausdorff spaces agree by 4.4. So since each $M(X_\alpha)$ is a dF space and hence complete by 6.4, and since the locally convex direct sum of complete spaces is complete; $M(\oplus X_\alpha)$ is complete. QED on 5)

6) X locally compact and paracompact implies that the hypothesis of 5) are satisfied. QED on 6)

7) If X is as in the hypothesis, then $c_c(X)$ is separable (cf. p.270 of [48]). Since $c_c(X)$ is barrelled, bounded subsets of $M(X) = [c_c(X)]^P$ are equicontinuous. By p. 252 of [19], equicontinuous subsets of the dual of a separable space are metrizable for the topology induced on them by the weak topology. But on equicontinuous sets, this topology agrees with the topology of precompact convergence. So because $M(X) = [c_c(X)]^P$, bounded sets are metrizable for the relative topology induced by $M(X)$. QED on 7)

QED

14.10 Theorem. Suppose X is a k -space. Let $M(X)$ and ϵ_X be as in 10.5. Then ϵ_X is a homeomorphism onto its range iff X is a Tychonoff space.

Proof: (\Rightarrow) $M(X)$ is a Tychonoff space, so the range of ϵ_X is Tychonoff with the relative topology. Hence X is Tychonoff.

(\Leftarrow) X Tychonoff implies that the points of X are closed. Hence if I can show that $\epsilon_X(x_\alpha) \rightarrow \epsilon_X(x_0)$ implies $x_\alpha \rightarrow x_0$, then ϵ_X will be injective and a homeomorphism onto its range (since we already know that ϵ_X is continuous). Suppose $\epsilon_X(x_\alpha) \rightarrow \epsilon_X(x_0)$. Let U be an open set in X such that $x_0 \in U$. Since X is completely regular, there exists a $f: X \rightarrow [0,1]$ such that $f(x_0) = 0$ and $f(x) = 1$ for all $x \notin U$.

$\epsilon_X(x_\alpha) \longrightarrow \epsilon_X(x_0)$ in $M(X)$ implies that
 $\epsilon_X(x_\alpha)(g) \longrightarrow \epsilon_X(x_0)(g)$ for all $g \in c(X)$. In particular
 $f(x_\alpha) \longrightarrow f(x_0)$. So there exists a β such that $\alpha \geq \beta$
implies $1/2 > |f(x_\alpha) - f(x_0)| = f(x_\alpha)$. Hence $\alpha \geq \beta$
implies that $x_\alpha \in U$, because if $\alpha \geq \beta$ and $x_\alpha \notin U$, then
 $f(x_\alpha) = 1$. Hence $x_\alpha \longrightarrow x_0$. QED

14.11 Theorem. Let X be a topologically p -complete k -space. Then

1) If $\mu \in M(X)$, then there exists a unique compact set K such that

a) if $f \in c(X)$ and $f(K) \subset \{0\}$, then
 $\mu(f) = 0$; and

b) if L is a closed subset of X such that
 $f \in c(X)$ and $\text{Supp } f \subset X \sim L$ implies $\mu(f) = 0$, then
 $L \supset K$.

2) The image of ϵ_X in $M(X)$ is closed.

14.12 Definition. If X satisfies the hypothesis of the last theorem and $\mu \in M(X)$, then call the compact set in part 1) of the theorem the support of μ and denote it by $\text{Supp } \mu$.

14.13 Theorem. Suppose X is a topologically p -complete k -space, then if $B \subset M(X)$ is bounded, there exists a compact set $K \subset X$ such that for all $\mu \in B$, $\text{Supp } \mu \subset K$.

Proof of 14.11 and 14.13: By 14.1, $M(X) = [c_c(X)]^p$.

So 1) of 14.11 follows from 13.5_^^{13.7} and 14.13 follows from 13.13. So all that remains is to show that the image of

ϵ_X is closed.

Let $\beta(X)$ be the Stone-Cech compactification of X . Regard X as a subspace of $\beta(X)$. Let i be the inclusion map from X to $\beta(X)$. By the definition of $M(i)$

$$\begin{array}{ccc} X & \xrightarrow{i} & \beta(X) \\ \epsilon_X \downarrow & & \downarrow \epsilon_{\beta(X)} \\ M(X) & \xrightarrow{M(i)} & M(\beta(X)) \end{array} \quad \text{commutes.}$$

Note that $c_c(X)$ and $c_c(\beta(X))$ are barrelled, hence p -determined; and that the restriction map

$a: c_c(\beta(X)) \rightarrow c_c(X)$ has dense image (cf. 11.5). This implies that $M(i) = {}^t a$ is injective.

Claim: $\epsilon_X(X) = M(i)^{-1}[\epsilon_{\beta(X)}(\beta(X))]$.

Proof of claim: Since the above diagram commutes, the \subset inclusion is obvious.

Suppose $\mu \in M(i)^{-1}(\epsilon_{\beta(X)}(\beta(X)))$. Then

$M(i)(\mu) = \epsilon_{\beta(X)}(q)$ for some $q \in \beta(X)$. Suppose $q \in \beta(X) \setminus X$. Now $\text{Supp } \mu$ is compact, hence closed in $\beta(X)$. Also $\text{Supp } \mu \subset X$, so $q \notin \text{Supp } \mu$. So there exists a $g: \beta(X) \rightarrow [0,1]$ such that $g(q) = 0$ and $g(x) = 1$ for all $x \in \text{Supp } \mu$. Let $\mathbb{1}: \beta(X) \rightarrow [0,1]$ be the constantly one function. Now $(g \circ i)(x) - (\mathbb{1} \circ i)(x) = 0$ for all $x \in \text{Supp } \mu$. So $\mu(\mathbb{1} \circ i) = \mu(g \circ i)$. So

$[M(i)(\mu)](g) = [M(i)(\mu)](\mathbb{1})$. Hence we have

$$\begin{aligned} 0 &= g(q) = \epsilon_{\beta(X)}(q)(g) = M(i)(\mu)(g) = M(i)(\mu)(\mathbb{1}) = \\ &= \epsilon_{\beta(X)}(q)(\mathbb{1}) = \mathbb{1}(q) = 1. \quad \text{So } 1 = 0. \quad \text{Contradiction.} \end{aligned}$$

Hence $q \in X$. But

$$M(i)(\epsilon_X(q)) = (M(i) \circ \epsilon_X)(q) = (\epsilon_{\beta(X)} \circ i)(q) = \epsilon_{\beta(X)}(q) =$$

$= M(i)(\mu)$. So since $M(i)$ is injective, $\varepsilon_X(q) = \mu$, where $q \in X$. So $\varepsilon_X(X) \supset M(i)^{-1}(\varepsilon_{\beta(X)}(\beta(X)))$.

QED on claim

But now we are essentially done because the continuous image of a compact set is compact; hence

$\varepsilon_{\beta(X)}(\beta(X))$ is compact, hence closed. But $M(i)$ is continuous, so the inverse image of a closed set is closed. Hence $\varepsilon_X(X)$ is closed. QED

14.14 Corollary. Let X be a topologically p -complete k -space. Let E be any locally convex Hausdorff space. Let $T: M(X) \rightarrow E$ be a linear map. Then T is continuous provided for all $K \subset X$ compact, T restricted to the closed linear subspace $\{\mu \in M(X) : \text{Supp } \mu \subset K\}$ is continuous.

Proof: By 13.10, the space in question is a closed linear subspace of $M(X)$.

Let $C \subset M(X)$ be compact and contain zero. Then by the last theorem, there exists a $K \subset X$ which is compact such that $C \subset \{\mu \in M(X) : \text{Supp } \mu \subset K\}$. Thus using the hypothesis, we find that $T|_C$ is continuous. Hence T is continuous, since $M(X)$ is a k -space by 14.9. QED

14.15 Proposition. Let X be a topologically p -complete k -space. Suppose I is a directed set, $\mu_\alpha \in M(X)$ for all $\alpha \in I$, and $\mu \in M(X)$. Then $\mu_\alpha \rightarrow \mu$ implies that $\text{Supp } \mu \subset [\cup\{\text{Supp } \mu_\alpha : \alpha \in C\}]^-$, where C is any

cofinal subset of I .

Proof: Suppose $\mu_\alpha \longrightarrow \mu$ in $M(X)$. Then $\mu_\alpha \longrightarrow \mu$ in $\sigma(c_c(X)', c_c(X))$ (cf. 14.1). So the result follows by 13.9. QED

14.16 Proposition. Let X be a topologically p -complete k -space. Then

- 1) If $\mu \in M(X)$, then $\mu = 0$ iff $\text{Supp } \mu = \emptyset$.
- 2) If λ is a non-zero scalar and $\mu, \nu \in M(X)$, then $\text{Supp } \lambda\mu = \text{Supp } \mu$ and $\text{Supp}(\nu+\mu) \subset \text{Supp } \nu \cup \text{Supp } \mu$.
- 3) If $A \subset X$, then $\{\mu \in M(X) : \text{Supp } \mu \subset A\}$ is a linear subspace of $M(X)$, and it is closed if A is closed in X .

4) Let A be a closed subspace of X . Suppose either A is compact or X is hemi-compact. Let $i: A \longrightarrow X$ be the inclusion map. Then $M(i)$ is an isomorphism in the category \mathcal{C} from $M(A)$ onto $\{\mu \in M(X) : \text{Supp } \mu \subset A\}$ where the latter space has the relative topology from $M(X)$.

Proof: 14.1, 13.8, 13.9, 13.10, and 13.11. QED

14.17 Proposition. Let X and Y be topologically p -complete k -spaces. Suppose $f: X \longrightarrow Y$ is continuous. Then for all $\mu \in M(X)$, $\text{Supp}(M(f)(\mu)) \subset f(\text{Supp } \mu)$.

Proof: Suppose $g \in c(Y)$ and $\text{Supp } g \subset Y \sim f(\text{Supp } \mu)$. Then $M(f)(\mu)(g) = \mu(g \circ f)$. Now

$$\text{Supp}(g \circ f) = [(g \circ f)^{-1}(\{r \in K : r \neq 0\})]^- = [f^{-1}(g^{-1}(\{r \in K : r \neq 0\}))]^-$$

$\subset f^{-1}(\text{Supp } g) \subset f^{-1}(Y \sim f(\text{Supp } \mu)) \subset X \sim \text{Supp } \mu$. So $\mu(g \circ f) = 0$. Thus $M(f)(\mu)(g) = 0$. So $\text{Supp } M(f)(\mu)$ is contained in $f(\text{Supp } \mu)$, since $f(\text{Supp } \mu)$ is closed.

QED

Most of the nice properties that the functor M enjoys result from the fact that M has a coadjoint, and hence preserves colimits. So M takes direct sums, direct limits, difference cokernels, and epics in \mathcal{J} , to the same thing in \mathcal{L} .

In general we cannot expect that M preserve limits, but in certain nice cases it does.

14.18 Theorem. Let X and Y be Tychonoff k -spaces and let X be topologically p -complete. If $f: X \rightarrow Y$ is injective and continuous, then $M(f): M(X) \rightarrow M(Y)$ is injective.

Proof: Let $a: c_c(Y) \rightarrow c_c(X)$ be defined by $a(r) = r \circ f$. Then the image of a is dense in $c_c(X)$ by 11.5. By 14.1, $M(X) = [c_c(X)]^P$. Suppose $\mu \in M(X)$ and $M(f)(\mu) = 0$. Then $M(f)(\mu)(r) = \mu(r \circ f) = \mu(a(r)) = 0$ for all $r \in c(Y)$. Thus by the density of the image of a , we find that $\mu = 0$. So $M(f)$ is injective. QED

14.19 Theorem. Let X and Y be k -spaces and let Y be Tychonoff. Suppose either that Y is hemi-compact or X is compact. Then if $f: X \rightarrow Y$ is a homeomorphism onto a closed subspace of Y , then $M(f): M(X) \rightarrow M(Y)$ is a \mathcal{C} -isomorphism onto a closed linear subspace of $M(Y)$.

Moreover $M(X)$ is complete.

Proof: Without loss of generality we may assume that X is a closed subspace of Y and f is the inclusion map. By theorem 11.2, there exists a continuous map σ from $c_c(X)$ to $c_c(Y)$ such that $\beta \circ \sigma = 1_{c_c(X)}$ where $\beta: c_c(Y) \rightarrow c_c(X)$ is the map $g \mapsto g \circ f$.

Suppose K is a compact set in $c_c(X)$, then $K = \beta(\sigma(K))$ and $\sigma(K)$ is a compact set in $c_c(Y)$. Now X is hemi-compact, so $c_c(X)$ is Frechet. Thus $M(X) = [c_c(X)]^{\mathbb{P}}$.

Suppose $M(f)(\mu_\alpha) \rightarrow 0$ in $M(Y)$. I would like to show that $\mu_\alpha \rightarrow 0$ uniformly on K . But $\sigma(K)$ is compact in $c_c(Y)$. Hence it is compact in $\alpha(c_c(Y))$ where α is the p -determination functor (cf. 2.1). So for all $\epsilon > 0$ there exists a β such that if $\alpha \geq \beta$ and $k \in K$, then

$\epsilon > |[M(f)(\mu_\alpha)](\sigma(k))| = |\mu_\alpha(\beta(\sigma(k)))| = |\mu_\alpha(k)|$. So $\mu_\alpha \rightarrow 0$ uniformly on K . Thus $\mu_\alpha \rightarrow 0$. Hence $M(f)$ is a topological linear isomorphism onto its range.

Now X is hemi-compact, so $M(X)$ is a dF space by 14.9. Hence $M(X)$ is complete. But a complete subspace of a Hausdorff uniform space is closed. So the range of $M(f)$ is closed and complete. QED

14.20 Theorem. Same hypothesis as 14.19, except assume in addition that Y is topologically p -complete. Then $M(f)$ is a \mathcal{C} -isomorphism from $M(X)$ onto $\{\mu \in M(Y) : \text{Supp } \mu \subset f(X)\}$ which is complete, hence

closed, in $M(Y)$.

Proof: Without loss of generality we may assume that X is a closed subset of Y and $f: X \rightarrow Y$ is the inclusion map. Then the result follows from 4) of 14.16. QED

Several more results of this sort will be given after the concept of the norm of an element of $M(X)$ is introduced.

Definition.

14.21 \wedge Let X be a topologically p -complete k -space. Then for all $\mu \in M(X)$ define a semi-norm p_μ on $c(X)$ as follows:

$$p_\mu(f) = \sup\{|f(x)| : x \in \text{Supp } \mu\}.$$

Also define a number $\|\mu\| \in [0, \infty]$ as follows:

$$\|\mu\| = \sup\{|\mu(f)| : f \in c(X) \text{ and } p_\mu(f) \leq 1\}.$$

14.22 Theorem. Let X be a topologically p -complete k -space. Then

$$0) \quad \|\mu\| = \sup\{|\mu(f)| : f \in c(X) \text{ and } |f(x)| \leq 1 \forall x \in X\}$$

for all $\mu \in M(X)$.

$$1) \quad \|\mu\| < \infty \text{ for all } \mu \in M(X) \text{ and } \|\cdot\| \text{ is a norm.}$$

$$2) \quad |\mu(f)| \leq \|\mu\| p_\mu(f) \text{ for all } f \in c(X) \text{ and } \mu \in M(X).$$

3) If $\mu \in M(X)$ and $f_\alpha \rightarrow f$ uniformly on $\text{Supp } \mu$, then $\mu(f_\alpha) \rightarrow \mu(f)$.

4) If E is a p -complete space, $f: X \rightarrow E$ is continuous, and $\bar{f}: M(X) \rightarrow E$ is the unique continuous linear map such that $\bar{f} \circ \epsilon_X = f$; then for all compact subsets K of X , $\bar{f}(\{\mu \in M(X) : \text{Supp } \mu \subset K \text{ \& } \|\mu\| \leq 1\})$ equals the closed, balanced, convex hull of $f(K)$.

5) a) If X is compact, then the norm defined above agrees with the usual norm on $M(X) = c_c(X)'$ and hence $(M(X), \|\cdot\|)$ is a Banach space.

b) Let K be a compact subset of X . If $i: K \rightarrow X$ denotes the inclusion map, then $M(i)$ regarded as a linear isomorphism from $M(K)$ onto $\{\mu \in M(X) : \text{Supp } \mu \subset K\}$ is an isometry, if both $\{\mu \in M(X) : \text{Supp } \mu \subset K\}$ and $M(K)$ are given the norm topologies (cf. 14.16). Hence for all $K \subset X$, compact, $\{\mu \in M(X) : \text{Supp } \mu \subset K\}$ is a Banach space and is thus norm closed in $M(X)$.

6) For all K compact, the inclusion map I_K , from $\{\mu \in M(X) : \text{Supp } \mu \subset K\}$ with the norm topology to $c_c(X)'$ with the topology of bounded convergence, is continuous.

7) The norm topology on $M(X)$ is finer than the $\sigma(c_c(X)', c_c(X))$ topology on $M(X)$ iff X is compact.

8) If $B \subset M(X)$ is $\sigma(c_c(X)', c_c(X))$ -bounded and is closed in $c_c(X)'$ with the topology of bounded convergence, then B is norm complete.

9) $\{\mu \in M(X) : \|\mu\| \leq 1\}$ is $\sigma(c_c(X)', c_c(X))$ -closed.

10) Let \mathcal{M} = the category of topologically p -complete, k -spaces and continuous maps. Let \mathcal{E} = the category of normed spaces and continuous linear maps of norm ≤ 1 . If for all $X \in \mathcal{M}$, $M_n(X)$ denotes $M(X)$ with the norm topology, then M_n is a functor from \mathcal{M} to \mathcal{E} , i.e. if X and Y are in \mathcal{M} and $f: X \rightarrow Y$ is continuous, then $\|M(f)(\mu)\| \leq \|\mu\|$ for all $\mu \in M(X)$.

Proof: Proof of 0). Let $\mu \in M(X)$. Define $p: c(X) \rightarrow [0, \infty]$ by $p(f) = \sup\{|f(x)| : x \in X\}$. Then for all $f \in c(X)$ and $\mu \in M(X)$, $p_\mu(f) \leq p(f)$. So $\|\mu\| \geq \sup\{|\mu(f)| : f \in c(X) \text{ and } p(f) \leq 1\}$.

Suppose $f \in c(X)$ and $p_\mu(f) \leq 1$. Then by the Tietze extension theorem and the Stone-Cech compactification theorem, we can extend $f|_{\text{Supp } \mu}$ to a $g \in c(X)$ such that $p(g) \leq 1$. Now $(g - f)|_{\text{Supp } \mu} = 0$, so $\mu(f) = \mu(g)$. Thus $|\mu(f)| \leq \sup\{|\mu(g)| : g \in c(X) \text{ \& } p(g) \leq 1\}$. Thus $\|\mu\| \leq \sup\{|\mu(g)| : g \in c(X) \text{ and } p(g) \leq 1\}$.

QED on 0)

14.22a Corollary to 0). If X is compact, then $\|\cdot\|$ is just the usual norm on $c_c(X)' = M(X)$. QED

14.22b Lemma. Let K be a compact subset of X . Let $i: K \rightarrow X$ be the inclusion map, then for all $\mu \in M(K)$, $\|\mu\|_K = \|M(i)(\mu)\|_X$.

Proof of lemma: Suppose $f \in c(K)$ and $|f(x)| \leq 1$ for all $x \in K$. Let g be any extension of f to X such that $|g(x)| \leq 1$ for all $x \in X$. Then $|\mu(f)| = |\mu(g \circ i)| = |[M(i)(\mu)](g)| \leq \|M(i)(\mu)\|$. So using 0) above (for $X = K$) we get $\|\mu\| \leq \|M(i)(\mu)\|$.

Suppose $g \in c(X)$ and $|g(x)| \leq 1$ for all $x \in K$. Then $\|\mu\| \geq |\mu(g \circ i)| = |[M(i)(\mu)](g)|$. So by 0) above, $\|\mu\| \geq \|M(i)(\mu)\|$. QED on lemma

Proof of 1). Let $\mu \in M(X)$. Let $K = \text{Supp } \mu$. K is compact. Let $i: K \rightarrow X$ be the inclusion map. Then by 14.16, $M(i): M(K) \rightarrow \{\mu \in M(X) : \text{Supp } \mu \subset K\}$ is a linear isomorphism. Thus $\|\mu\|_X = \| [M(i)]^{-1}(\mu) \|_K$ by the last

lemma. But by 14.22a, $\|M(i)^{-1}(\mu)\|_K$ is just the usual norm of $M(i)^{-1}(\mu)$ in $c_c(K)' = M(K)$. Hence it is less than infinity. Hence $\|\mu\| < \infty$, for all $\mu \in M(X)$.

Obviously if $\mu = 0$, then $\|\mu\| = 0$.

Claim: $\|\mu\| = 0$ implies $\mu = 0$.

Proof of claim: Suppose $\mu \neq 0$. Then there exists a $f \in c(X)$ such that $\mu(f) \neq 0$. But then $p_\mu(f) > 0$. So $0 < |\mu(f)/p_\mu(f)| = |\mu(f/p_\mu(f))| \leq \|\mu\|$. QED on claim

Each of the other norm axioms follow trivially from 0). QED on 1)

Proof of 5). a) of 5) follows immediately from 0). b) of 5) follows immediately from 14.22b.

Proof of 2). Let $\mu \in M(X)$ and $f \in c(X)$. Let $K =$ the support of μ . Let $i: K \rightarrow X$ be the inclusion map. Then by b) of 5), $\|\mu\|_X = \|M(i)^{-1}(\mu)\|_K$. Using a) of 5) we get that $|M(i)^{-1}(\mu)(f|K)| \leq p_\mu(f) \|M(i)^{-1}(\mu)\|_K$. Now if $g \in c(K)$, then $[M(i)^{-1}(\mu)](g) = \mu(r)$ where r is any extension of g to all of X (cf. 14.11). So $|[M(i)^{-1}(\mu)](f|K)| = |\mu(f)|$. So we see that

$$|\mu(f)| \leq p_\mu(f) \|\mu\|. \quad \text{QED on 2)}$$

Proof of 3). 3) follows from 2) together with the fact that μ is linear and $\|\mu\| < \infty$. QED on 3)

Proof of 6). Let $q: c_c(X) \rightarrow \mathbb{R}^+$ be the continuous semi-norm defined by $q(f) = \sup\{|f(x)| : x \in K\}$. If $\text{Supp } \mu \subset K$, then $p_\mu \leq q$. So for all $f \in c(X)$ and for all $\mu \in \{\mu \in M(X) : \text{Supp } \mu \subset K\}$, we get using 2)

$|\mu(f)| \leq \|\mu\| q(f)$. Let B be a bounded subset of $c_c(X)$ and $\varepsilon > 0$. Then there exists a $M > 0$ such that

$q(b) \leq M$ for all $b \in B$. This is true since continuous semi-norms take bounded sets to bounded sets. Thus if $\mu \in M(X)$, $\text{Supp } \mu \subset K$, $\|\mu\| < \varepsilon/M$, and $b \in B$, we have $|\mu(b)| \leq \|\mu\| q(b) \leq [\varepsilon/M]M = \varepsilon$. So I_K is continuous into $c_c(X)'$ with bounded convergence. QED on 6)

Proof of 8). Suppose $B \subset M(X)$ is as in the hypothesis. Then by 14.13, there exists a $K \subset X$, compact, such that $B \subset \{\mu \in M(X) : \text{Supp } \mu \subset K\}$. The continuity of I_K (from 6 above) says that B is norm closed in there. But that space is norm complete by 5). So B is norm complete. QED on 8)

Proof of 9). Suppose $\mu_\alpha \rightarrow \mu$ in $\sigma(c_c(X)', c_c(X))$ and $\|\mu_\alpha\| \leq 1$ for all α . Let $f \in c(X)$ be such that $|f(x)| \leq 1$ for all $x \in X$. Then $|\mu_\alpha(f)| \leq \|\mu_\alpha\| p_{\mu_\alpha}(f) \leq 1$ for all α . So since $|\mu_\alpha(f)| \rightarrow |\mu(f)|$, we have $|\mu(f)| \leq 1$. So $\sup\{|\mu(f)| : p(f) \leq 1\} \leq 1$. So $\|\mu\| \leq 1$ by 0). Hence the set is $\sigma(c_c(X)', c_c(X))$ -closed.

QED on 9)

Proof of 7). If X is compact, then by 5) we see that the norm is the same as the usual norm on $c_c(X)' = M(X)$. So the norm topology is just uniform convergence on bounded sets of $c_c(X)$, and hence it is finer than the $\sigma(c_c(X)', c_c(X))$ topology.

On the other hand, if $M(X)$ with the norm topology is finer than the $\sigma(c_c(X)', c_c(X))$ topology, then since $\{\varepsilon(x) : x \in X\}$ is norm bounded it would be $\sigma(c_c(X)', c_c(X))$ bounded and hence bounded for the usual topology on $M(X)$ by Mackey's theorem. But the image of ε is closed

in $M(X)$ with its usual topology by 14.11. So $\varepsilon(X)$ is closed and bounded. Hence $\varepsilon(X)$ is compact, since $M(X)$ is a semi-Montel space by 14.9. But ε is a homeomorphism by 14.10. So X is compact. QED on 7)

Proof of 4). Let K , f , and E be as in the hypothesis. $\{\mu \in M(X) : \|\mu\| \leq 1\}$ is closed, convex, and balanced in $M(X)$ by 9). $\{\mu \in M(X) : \text{Supp } \mu \subset K\}$ is closed, convex, and balanced by 14.16. Let $A = \{\mu \in M(X) : \text{Supp } \mu \subset K \text{ and } \|\mu\| \leq 1\}$. Then A is closed, convex, and balanced in $M(X)$. But by 6), A is also bounded in $M(X)$, since it is bounded in $\{\mu \in M(X) : \text{Supp } \mu \subset K\}$ with the norm topology. But closed and bounded implies compact, hence A is compact in $M(X)$. So $\bar{f}(A)$ is compact, hence closed in E . It is also balanced and convex.

Now for all $x \in K$, $\|\varepsilon(x)\| \leq 1$, $\text{Supp } \varepsilon(x) \subset K$, and $\bar{f}(\varepsilon(x)) = f(x)$. Thus we see that $f(K) \subset \bar{f}(A)$. And so the closed, convex, balanced hull of $f(K)$ is also contained in $\bar{f}(A)$. So we are half done.

Let $\mu \in A$ and $\phi \in f(K)^\circ \subset E'$. Then by 2) we get $|\bar{f}(\mu)(\phi)| = |\mu(\phi \circ f)| \leq \|\mu\| p_\mu(\phi \circ f) \leq p_\mu(\phi \circ f)$. But $p_\mu(\phi \circ f) = \sup\{|\phi(f(x))| : x \in \text{Supp } \mu\} \leq 1$, since $\text{Supp } \mu \subset K$ and $\phi \in f(K)^\circ$. Thus $|\bar{f}(\mu)(\phi)| \leq 1$. So $\bar{f}(\mu) \in f(K)^{\circ\circ} =$ the closed, convex, balanced hull of $f(K)$. So $\bar{f}(A) =$ the closed, convex, balanced hull of $f(K)$.

QED on 4)

Proof of 10): We already know that M is a functor into the category of vector spaces. Then only thing to check is that if X and Y are in \mathcal{M} and $f: X \rightarrow Y$ is

continuous, then $\|M(f)(\mu)\| \leq \|\mu\|$ for all $\mu \in M(X)$.

Let $h \in c(Y)$ such that $|h(y)| \leq 1$ for all $y \in Y$.

Then $|(h \circ f)(x)| \leq 1$ for all $x \in X$, so

$$|M(f)(\mu)(h)| = |\mu(h \circ f)| \leq \|\mu\|. \text{ So by 0), } \|M(f)(\mu)\| \leq$$

$\|\mu\|$.

QED on 10)

QED

14.23 Theorem. If X is locally compact and topologically p -complete. Let $c_\infty(X)$ denote all continuous scalar-valued functions on X which vanish at infinity with the supremum norm. Let $M_b(X)$ denote the Banach space dual of $c_\infty(X)$. Let $a: c_\infty(X) \rightarrow c_c(X)$ be inclusion. Let Supp_b denote the support of a measure in $M_b(X)$. Then

1) ${}^t a: M(X) \rightarrow M_b(X)$ is an isometry onto $\{\mu \in M_b(X) : \text{Supp}_b \mu \text{ is compact}\}$ if both spaces are given their respective norms; and

2) $\text{Supp}_b [{}^t a(\mu)] = \text{Supp } \mu$ for all $\mu \in M(X)$.

Proof: Suppose $\mu \in M(X)$ and $f \in c_\infty(X)$ such that

$|f(x)| \leq 1$ for all $x \in X$. Then

$$|{}^t a(\mu)(f)| = |\mu(f)| \leq \|\mu\|_{M(X)}. \text{ So } \|{}^t a(\mu)\| \leq \|\mu\|.$$

Let $\mu \in M(X)$ and $f \in c(X)$ such that $|f(x)| \leq 1$ for all $x \in X$. Now X is locally compact. So $\text{Supp } \mu$ has a compact neighborhood K because $\text{Supp } \mu$ is compact. Now there exists a $g \in c(X)$ such that

$\text{Supp } g \subset K$, $|g(x)| \leq 1$ for all $x \in X$, and $g(t) = 1$ for all $t \in \text{Supp } \mu$. So $fg \in c_\infty(X)$ and $fg(x) = f(x)$

for all $x \in \text{Supp } \mu$. Hence $\mu(fg) = \mu(f)$. Thus

$$|\mu(f)| = |\mu(fg)| = |{}^t a(\mu)(fg)| \leq \|{}^t a(\mu)\| \text{ since}$$

$|f(x)g(x)| \leq 1$ for all $x \in X$. So $\|\mu\| \leq \|{}^t_a(\mu)\|$ by 0) of 14.22. Hence $\|\mu\| = \|{}^t_a(\mu)\|$. So t_a is an isometry.

Let $\mu \in M(X)$.

I would like to show that $\text{Supp } \mu = \text{Supp}_b {}^t_a(\mu)$.

Since $c_\infty(X) \subset c(X)$, it is trivial that

$\text{Supp}_b {}^t_a(\mu) \subset \text{Supp } \mu$.

Suppose $f \in c(X)$ and $\text{Supp } f \subset X \sim \text{Supp}_b {}^t_a(\mu)$.

$\text{Supp } \mu$ is compact. So as above there exists a $g \in c(X)$

with compact support such that $g(x) = 1$ for all

$x \in \text{Supp } \mu$. So $fg(x) = f(x)$ for all $x \in \text{Supp } \mu$. Hence

$\mu(fg) = \mu(f)$ and $\text{Supp } fg \subset \text{Supp } f \cap \text{Supp } g$. So fg

has compact support. Now ${}^t_a(\mu)(fg) = 0$ since

$\text{Supp } fg \subset \text{Supp } f \subset X \sim \text{Supp}_b {}^t_a(\mu)$. But

${}^t_a(\mu)(fg) = \mu(fg) = \mu(f)$. So $\text{Supp } f \subset X \sim \text{Supp}_b {}^t_a(\mu)$

implies $\mu(f) = 0$. Thus $\text{Supp } \mu \subset \text{Supp } {}^t_a(\mu)$.

Hence $\text{Supp } \mu = \text{Supp } {}^t_a(\mu)$.

Thus we have shown that t_a maps $M(X)$ into

$\{\mu \in M_b(X) : \text{Supp}_b \mu \text{ is compact}\}$. It only remains to be shown that it is onto this space.

Suppose $\nu \in M_b(X)$ and $\text{Supp}_b \nu$ is compact. Now the concept of support in $M_b(X)$ has the properties:

1) if $g, f \in c_\infty(X)$ and $f(x) = g(x)$ for all $x \in \text{Supp}_b \nu$, then $\nu(f) = \nu(g)$; and

2) if $f_\alpha \rightarrow 0$ uniformly on $\text{Supp}_b \nu$, then $\nu(f_\alpha) \rightarrow 0$.

Let $g \in c(X)$ such that g has compact support and $g(x) = 1$ for all $x \in \text{Supp}_b \nu$. Define $\mu \in M(X)$ by $\mu(f) = \nu(fg)$ for all $f \in c(X)$ (μ is continuous

on $c_c(X)$ because of 2) immediately above). Now suppose $h \in c_\infty(X)$. Then since $h(x)g(x) = h(x)$ for all $x \in \text{Supp}_D v$, we have $t_a(\mu)(h) = \mu(h) = v(hg) = v(h)$. So $t_a(\mu) = v$. QED

The following seems to be the best that I can say about generalizing 14.20.

14.24 Theorem. Let X and Y be topologically p -complete k -spaces. Suppose $f: X \rightarrow Y$ is a homeomorphism onto its range. Then $M(f)$ is a continuous injective map from $M(X)$ onto $\{v \in M(Y) : \text{Supp } v \subset f(X)\}$.

Proof: That $M(f)$ is injective follows from 14.18. The inclusion \subset follows from 14.17.

Suppose $v \in M(Y)$ and $\text{Supp } v \subset f(X)$. Throughout this proof, if $f: \text{Supp } v \rightarrow \mathbb{K}$ is a continuous map, let \bar{f} denote an extension of f to all of Y . This extension exists since $\text{Supp } v$ is compact and Y is Tychonoff.

Let $q: \text{Supp } v \rightarrow X$ be defined by $q(x) = f^{-1}(x)$. Define $\bar{v} \in M(X)$ by $\bar{v}(g) = v(\bar{g} \circ q)$. \bar{v} is well-defined by 14.11 and continuous by 3) of 14.22.

Let $h \in c(Y)$. Then $M(f)(\bar{v})(h) = \bar{v}(h \circ f) = v(\bar{h} \circ \bar{f} \circ q)$. But $y \in \text{Supp } v$ implies $\bar{h} \circ \bar{f} \circ q(y) = h \circ f \circ q(y) = h(f(f^{-1}(y))) = h(y)$. So by 14.11, $v(\bar{h} \circ \bar{f} \circ q) = v(h)$. Thus $M(f)(\bar{v})(h) = v(h)$. Hence $M(f)(\bar{v}) = v$. So v is in the range of $M(f)$. QED

14.25 Theorem. Let X be a topologically p -complete k -space. Let $\beta(X)$ be the Stone-Cech compactification

of X . Let $i: X \rightarrow \beta(X)$ be the inclusion map. Give $M(X)$ and $M(\beta(X))$ the norms defined in 14.21. Then $M(i)$ is an isometric isomorphism of $M(X)$ onto $\{\nu \in M(\beta(X)) : \text{Supp } \nu \subset X\}$.

Proof: 14.24 and 10) of 14.22 proves everything except that if $\mu \in M(X)$, then $\|\mu\| \leq \|M(i)(\mu)\|$.

Let $\mu \in M(X)$ and $f \in c(X)$ such that $|f(x)| \leq 1$ for all $x \in X$. Then we can extend f to $\bar{f} \in c(\beta(X))$ such that $|\bar{f}(x)| \leq 1$ for all $x \in \beta(X)$. So $\|M(i)(\mu)\| \geq |M(i)(\mu)(\bar{f})| = |\mu(\bar{f} \circ i)| = |\mu(f)|$. Thus $\|\mu\| \leq \|M(i)(\mu)\|$ by 0) of 14.22. QED

14.26 Corollary. If X is a topologically p -complete k -space, then $M(X)$ is norm complete iff X is compact.

Proof: The (\Leftarrow) direction is proved in 14.22.

(\Rightarrow) Assume that X is not compact. Then since X is topologically p -complete, X can not be topologically precompact in X . Hence there exists a countably infinite discrete collection $\{U_j : j \in \omega\}$ of non-empty open sets such that $i \neq j$ implies $U_i \cap U_j = \emptyset$. For all $i \in \omega$, let $x_i \in U_i$. Let $\beta(X)$ be the Stone-Cech compactification of X . Consider X as a subspace of $\beta(X)$. Let $\varepsilon: \beta(X) \rightarrow M(\beta(X))$ be the canonical map. For all $i \in \omega$, let $\mu_i = \sum_{j=1}^i j^{-2} \varepsilon(x_j) \in M(\beta(X))$. Now if $j \leq k$, then $\|\mu_k - \mu_j\| = \|\sum_{i=j+1}^k i^{-2} \varepsilon(x_i)\| \leq \sum_{i=j+1}^k i^{-2}$, since $\|\varepsilon(x_i)\| = 1$ for all i . Hence $\{\mu_k\}$ is a norm Cauchy sequence. Now $M(\beta(X))$ is norm complete. So

there exists a $\mu \in M(\beta(X))$ such that $\mu_n \rightarrow \mu$ in norm.

Claim: $\text{Supp } \mu \not\subset X$ and for all i , $\text{Supp } \mu_i \subset X$.

Proof of claim: Note that $\text{Supp } \mu_i \subset \{x_j : 1 \leq j \leq i\} \subset X$ by 14.16 since for all i , $\text{Supp } \varepsilon(x_i) = \{x_i\}$.

By 7) of 14.22, $\mu_n \rightarrow \mu$ in $\sigma(c_c(\beta(X)), c_c(\beta(X)))$. So $\text{Supp } \mu \subset [\cup\{\text{Supp } \mu_i : i \in \omega\}]^- \subset \{x_i : i \in \omega\}^-$ by 14.15

(note that the closures are taken in $\beta(X)$). Suppose

$\text{Supp } \mu \subset X$. Then $\text{Supp } \mu \subset \{x_i : i \in \omega\}^- \cap X$. But

$\{x_i : i \in \omega\}^- \cap X$ is the closure in X of

$\{x_i : i \in \omega\} = \cup\{\{x_i\} : i \in \omega\}$. But this set is closed

in X , since $\{\{x_i\} : i \in \omega\}$ is locally finite (recall

where the x_i came from). So $\{x_i : i \in \omega\}^- \cap X =$

$\{x_i : i \in \omega\}$. Thus $\text{Supp } \mu \subset \{x_i : i \in \omega\} \subset \cup\{U_i : i \in \omega\}$.

Now $\text{Supp } \mu$ is a compact subset of X and the U_j are open in X , so there exists a $j \in \omega$ such that

$\text{Supp } \mu \subset \cup\{U_i : 1 \leq i \leq j\}$. By complete regularity, there

exists a continuous function $f: X \rightarrow [0,1]$ such that

$f(\text{Supp } \mu) \subset \{0\}$ and $f(x) = 1$ for all $x \notin \cup\{U_i : 1 \leq i \leq j\}$.

Since $\{U_i : i \in \omega\}$ is a discrete family, for all $k \geq j+1$

$x_k \notin \cup\{U_i : 1 \leq i \leq j\}$. So $f(x_k) = 1$ for all $k \geq j+1$.

Now extend f to a continuous function $\bar{f}: \beta(X) \rightarrow [0,1]$.

Now by 0) of 14.22, $\|\mu - \mu_n\| \geq |\mu(\bar{f}) - \mu_n(\bar{f})|$. Also

$\bar{f}(\text{Supp } \mu) = f(\text{Supp } \mu) \subset \{0\}$, so $\mu(\bar{f}) = 0$.

$\mu_n(\bar{f}) = \sum_{i=1}^n i^{-2} \varepsilon(x_i)(\bar{f}) = \sum_{i=1}^n i^{-2} \bar{f}(x_i) \geq \sum_{i=j+1}^n i^{-2}$.

So $\|\mu - \mu_n\| \geq \sum_{i=j+1}^n i^{-2} \geq (j+1)^{-2}$ provided $n \geq j+1$.

So μ_n cannot converge to μ . Contradiction. Hence it

is not true that $\text{Supp } \mu \subset X$.

QED on claim

Since $\mu_i \rightarrow \mu$, the claim demonstrates that $\{\mu \in M(\beta(X)) : \text{Supp } \mu \subset X\}$ is not norm closed in $M(\beta(X))$. Thus by 14.25, $M(X)$ with the norm topology cannot be complete. QED

14.27 Theorem. Suppose X and Y are k -spaces such that X is metrizable and Y is a Tychonoff space. Suppose $X \neq \emptyset$. Suppose $f: X \rightarrow Y$ is a homeomorphism onto a closed subspace of Y . Further suppose that at least one of following conditions hold:

- 1) Y is paracompact; and X is metrically topologically complete or $f(X)$ is a G_δ in Y .
- 2) Y is normal; X is separable; and X is metrically topologically complete or $f(X)$ is a G_δ in Y .
- 3) X is compact.

Then

- a) $M(f): M(X) \rightarrow M(Y)$ is a \mathcal{C} -isomorphism onto a closed linear subspace of $M(Y)$;
- b) there exists a surjective continuous open linear map $\phi: M(Y) \rightarrow M(X)$ such that $\phi \circ M(f) = I_{M(X)}$; and
- c) $M(Y) = (\text{kernel } \phi) \oplus (\text{Image } M(f))$ in the category of locally convex Hausdorff spaces.

And if in addition Y is topologically p -complete and M_n is as in (10) of 14.22, then

- a') $M_n(f): M_n(X) \rightarrow M_n(Y)$ is an isometric isomorphism onto a norm closed linear subspace of $M_n(Y)$;
- b') There exists a surjective continuous open

linear map $\phi: M_n(Y) \rightarrow M_n(X)$ of norm one such that $\phi \circ M_n(f) = 1_{M_n(X)}$; and

c') $M_n(f) \circ \phi$ is a projection of norm one from $M_n(Y)$ onto the image of $M_n(f)$.

14.28 Remark. If Y is topologically p -complete, then $\text{Image } M(f) = \text{Image } M_n(f) = \{v \in M(Y) : \text{Supp } v \subset f(X)\}$. This is due to 14.24.

14.29 Corollary. If Y is metrizable, $X \neq \emptyset$, and $f: X \rightarrow Y$ is a homeomorphism onto a closed subspace of Y , then a), b), c), a'), b'), and c') above each hold.

Proof of 14.29: Y metrizable implies that Y is topologically p -complete, and also that Y is paracompact and $f(X)$ is a G_δ in Y . Hence condition 1) above holds. QED on 14.29

Proof of 14.27: Without loss of generality we may assume that X is a closed subspace of Y and $f: X \rightarrow Y$ is the inclusion map. From 11.10 by taking E equal to the scalars, we see that if $\psi: c_c(Y) \rightarrow c_c(X)$ is the restriction map, then there exists a continuous linear map $\xi: c_c(X) \rightarrow c_c(Y)$ such that for all $f \in c(X)$ $\text{Range } \xi(f) \subset \text{convex hull}(\text{Range } f)$ and such that $\psi \circ \xi = 1_{c_c(X)}$. Operating on the last equation, first by the functor α (cf. 2.14) and then by the functor R (cf. 3.2), we find that ${}^t[\alpha(\xi)] \circ {}^t[\alpha(\psi)] = 1_{M(X)}$. But ${}^t[\alpha(\psi)] = M(f)$. So if we define $\phi = {}^t[\alpha(\xi)]$, then ϕ is a continuous linear map from $M(Y)$ to $M(X)$ such

that $\phi \circ M(f) = 1_{M(X)}$.

Now suppose $f \in c(X)$ and $|f(x)| \leq 1$ for all $x \in X$. Then $\text{Range } \xi(f) \subset \text{convex hull } (\text{Range } f) \subset \{t \in \mathbb{K} : |t| \leq 1\}$. Thus if $\mu \in M(Y)$

$$|\phi(\mu)(f)| = |\mu(\xi(f))| \leq \|\mu\|,$$

by the definition of the norm. So by 0) of 14.22, $\|\phi(\mu)\| \leq \|\mu\|$. So ϕ is a continuous map from $M_n(Y)$ to $M_n(X)$ such that $\|\phi\| \leq 1$

and $\phi \circ M_n(f) = 1_{M_n(X)}$. Also by 10) of 14.22, $\|M_n(f)\| \leq 1$.

So now for both a, b, c , and a', b' , and c' , apply 11.8 and 11.9. QED

14.30 Lemma. There exists a compact Hausdorff space X and a closed subset $A \subset X$ such that there does not exist a continuous linear map $\phi: M(X) \rightarrow M(A)$ with the property that $\phi \circ M(i) = 1_{M(A)}$, where $i: A \rightarrow X$ is the inclusion map.

Proof: Let X and A be as in 11.12. Suppose such a ϕ does exist. Since $c_c(X)$ and $c_c(A)$ are barrelled (cf. 13.14), the precompact sets of $c_c(X)^P$ (resp., $c_c(A)^P$) agree with the equicontinuous sets of $c_c(X)'$ (resp., $c_c(A)'$). Since ϕ and $M(i)$ send precompact sets to precompact sets, ${}^t\phi$ and ${}^tM(i)$ are continuous between $[M(X)]^P = c_c(X)$ and $[M(A)]^P = c_c(A)$. Now ${}^t[M(i)]$ is just the restriction map from $c_c(X)$ to $c_c(A)$. Thus since $1_{c_c(A)} = {}^t(1_{M(A)}) = {}^t(\phi \circ M(i)) = {}^tM(i) \circ {}^t\phi$, we have a contradiction to 11.12. QED

Section 15 - Approximation and nuclearity.

In this section sufficient conditions are given for $M(X)$ to satisfy the "approximation property". Also necessary and sufficient conditions are given for $M(X)$ to be a nuclear space.

The following proposition is the basis for the next theorem.

15.1 Proposition. Let X be a Tychonoff space and E be a locally convex Hausdorff space. Then the algebraic tensor product of $c(X)$ and E is dense in $c_c(X, E)$.

Proof:

By prop. 40.2 of [46], the continuous functions with finite dimensional range are the same as those functions in $c(X, E)$ which can be represented in the form $x \mapsto \sum_{i=1}^n f_i(x)e_i$, where each f_i is in $c(X)$ and each e_i is in E ; and furthermore these functions are an algebraic tensor product of $c(X)$ and E .

Suppose K is compact in X and U is a convex neighborhood of zero in E . Suppose $f \in c(X, E)$. I would like to find a function g with finite dimensional range such that $f(x) - g(x) \in U$ for all $x \in K$. Since

f is continuous, for all $x \in K$ there exists an open set V_x which contains x and such that $r \in V_x$ implies $f(r) - f(x) \in U$. Thus $\{V_x : x \in K\}$ is an open covering of K . Let $\{V_{x_1}, \dots, V_{x_n}\}$ be a finite subcover of K . Then by 13.1, there are g_1, \dots, g_n in $c(X)$ such that for all $1 \leq i \leq n$, $0 \leq g_i$; $\sum_{i=1}^n g_i(x) = 1$ for all $x \in K$; $\sum_{i=1}^n g_i(x) \leq 1$ for all $x \in X$; and $\text{Supp } g_i \subset V_{x_i}$ for all $1 \leq i \leq n$.

Now define \bar{f} by $\bar{f}(x) = \sum_{i=1}^n g_i(x) f(x_i)$ for all $x \in X$. Thus if $x \in K$,

$$\begin{aligned} f(x) - \bar{f}(x) &= [\sum_{i=1}^n g_i(x)] f(x) - [\sum_{i=1}^n g_i(x) f(x_i)] \\ &= \sum_{i=1}^n [g_i(x) (f(x) - f(x_i))] \end{aligned}$$

The terms where $g_i(x) = 0$ may be ignored. Suppose $g_i(x) \neq 0$. Then $x \in \text{Supp } g_i \subset V_{x_i}$. So $f(x) - f(x_i) \in U$. Hence if $x \in K$, then $f(x) - \bar{f}(x)$ is a convex combination of elements of U . But U is a convex set, so $f(x) - \bar{f}(x) \in U$ if $x \in K$. But \bar{f} has finite dimensional range. Hence the algebraic tensor product of $c(X)$ and E is dense in $c_c(X, E)$. QED.

15.2 Theorem. Let X be a topologically p -complete

k -space. Then $M(X)$ has the approximation property.

Proof: In this proof, \otimes will denote the algebraic tensor product. We will begin with a lemma.

15.2a Lemma. If E is a p -complete space, X is a k -space, and $f: X \rightarrow E$ is a continuous function of the form $f(x) = \sum_{i=1}^n f_i(x) e_i$, where each f_i is in $c(X)$

and each e_i is in E , then the unique continuous linear map \bar{f} from $M(X)$ to E such that $\bar{f} \circ \epsilon = f$ is defined by $\bar{f}(\mu) = \sum_{i=1}^n \mu(f_i)e_i$.

Proof of lemma: Let \bar{f} be defined by the equation immediately above. \bar{f} is certainly continuous, since each $\{f_i\}$ is precompact in $C(X)$. Also

$$(\bar{f} \circ \epsilon)(x) = \sum_{i=1}^n [\epsilon(x)(f_i)]e_i = \sum_{i=1}^n f_i(x)e_i = f(x).$$

So by uniqueness, \bar{f} fills the bill. QED on lemma

To continue with the proof of the theorem, by 15.1 $c_c(X) \otimes M(X)$ is dense in $c_c(X, M(X))$. But by 14.3, $c_c(X, M(X))$ is isomorphic to $\text{hom}_p(M(X), M(X))$. Hence by the lemma and the density property, we have that the set of $g \in \text{hom}(M(X), M(X))$ which are of the form $g(v) = \sum_{i=1}^n v(f_i)\mu_i$ where each f_i is in $c(X)$ and each μ_i is in $M(X)$, is dense in $\text{hom}_p(M(X), M(X))$.

But this set is just the canonical embedding of $[M(X)]' \otimes M(X)$ in $\text{hom}(M(X), M(X))$, if one remembers that $[M(X)]' = c(X)$. So $[M(X)]' \otimes M(X)$ is dense in $\text{hom}_p(M(X), M(X))$. Thus by pages 4 and 8 of expose 14 of [42], we see that $M(X)$ satisfies the approximation property. QED

15.3 Theorem. Let X be a Tychonoff k -space. Then $M(X)$ is a nuclear space iff X is a countable discrete space.

Proof: (\Rightarrow) Suppose $M(X)$ is nuclear. Let K be a compact subset of X . Let $i: K \rightarrow X$ be the inclusion

map. Then by 14.19, $M(i): M(K) \rightarrow M(X)$ is an isomorphism onto its range and its range is closed. Since a subspace of a nuclear space is again nuclear, $M(K)$ is nuclear. Now $c_c(K)$ is a Banach space and $M(K) = c_c(K)^P$. Thus $c_c(K)$ is nuclear by 8.11; and hence by theorem 9 on page 273 of [48], K must be a finite set. So all compact subsets of X are finite.

Claim: X has the discrete topology.

Proof of claim: Let $x \in X$. I would like to show that $\{x\}$ is an open set. Let K be a compact set. Then $(X \sim \{x\}) \cap K = K \sim \{x\}$ is a finite set since K is finite. But finite sets are closed in a Hausdorff space. So for all K compact, $(X \sim \{x\}) \cap K$ is closed. Hence because X is a k -space, $X \sim \{x\}$ is closed. Hence $\{x\}$ is open. Hence X has the discrete topology.

QED on claim

Thus I have shown that $M(X)$ nuclear implies that X has the discrete topology. I will complete the proof of (\Rightarrow) and will in addition prove (\Leftarrow) , if I can prove the following

Claim: If X is discrete, then $M(X)$ is nuclear iff X is countable.

Proof of claim: X discrete implies $X = \bigoplus \{\{x\} : x \in X\}$. So by 10.6, M preserves colimits, since it has a coadjoint. Thus $M(X) = \bigoplus \{M(\{x\}) : x \in X\}$. Now for all $x \in X$ $M(\{x\})$ is isomorphic to the scalars. An uncountable direct sum of scalars is not nuclear by page 6 of expose 17 of [42]. But a countable direct sum of scalars is

nuclear by prop. 50.1 (6) of [46].

QED

Section 16 - $\text{Hom}(M(X \boxtimes Y), E) \cong \text{Hom}(M(X), \text{Hom}(M(Y), E))$

Once we learn about tensor products in the next chapter, the above result will give an easy proof that M commutes with "tensor" products.

16.0 Definition. Let $\boxtimes: \mathcal{J} \times \mathcal{J} \rightarrow \mathcal{J}$ denote the product functor in the category \mathcal{J} , i.e. $X \boxtimes Y = k(X \times Y)$ where k is the "k-ification" functor (cf. C10).

16.1 Theorem. The functors

$$(X, Y, E) \mapsto c_C(X \boxtimes Y, E) \quad \text{and}$$

$$(X, Y, E) \mapsto c_C(X, c_C(Y, E)) \quad \text{from}$$

$\mathcal{J}^{\text{OP}} \times \mathcal{J}^{\text{OP}} \times \mathcal{C}$ to \mathcal{C} (respectively, from $\mathcal{J}^{\text{OP}} \times \mathcal{J}^{\text{OP}} \times \mathcal{B}$ to \mathcal{B}) are naturally isomorphic.

Proof: Let X and Y be k -spaces and let E be a locally convex Hausdorff space. Let $f \in c(X \boxtimes Y, E)$. Let $x \in X$. Let K be compact in Y . Let $y_\alpha \rightarrow y_0$ be a convergent net in K . Since $f \in c(X \boxtimes Y, E)$, $f|_{\{x\} \times K}$ is continuous (cf. C4). So because $(x, y_\alpha) \rightarrow (x, y_0)$, $f(x, y_\alpha) \rightarrow f(x, y_0)$. So $f(x, \cdot)$ restricted to each compact set of continuous. Thus since Y is a k -space, $f(x, \cdot) \in c(Y, E)$. Define $\mu(f): X \rightarrow c_C(Y, E)$ by $\mu(f)(x)(y) = f(x, y)$.

Claim: $\mu(f)$ is continuous.

Proof of claim: Let K be a compact subset of X .

It is sufficient to prove that $\mu(f)|_K$ is continuous,

since X is a k -space. Let L be compact in Y and U be open in E . [Recall that the topology of the uniformity of compact convergence and the compact-open topology agree.] Let $x \in K$. Suppose $\mu(f)(x) \in \{g \in c(Y, E) : g(L) \subset U\}$, i.e. $f(\{x\} \times L) \subset U$. Since $K \times L \rightarrow X \times Y$ is continuous and $f: X \times Y \rightarrow E$ is continuous, $[f|_{K \times L}]^{-1}(U)$ is open in $K \times L$. Now $\{x\} \times L$ is compact and $\{x\} \times L \subset [f|_{K \times L}]^{-1}(U)$, so there exists an open set N in Y containing x such that $\{x\} \times L \subset N \times L \subset [f|_{K \times L}]^{-1}(U)$ (cf. p.142 of [21]). So $\mu(f)(N) \subset \{g \in c(Y, E) : g(L) \subset U\}$. So $\mu(f)$ is continuous from X to $c_c(Y, E)$.

QED on claim

By C4, if K is compact in X and L is compact in Y , then $K \times L$ is compact in $X \times Y$. Hence by considering neighborhoods of zero we see that the canonical map μ is continuous from $c_c(X \times Y, E)$ to $c_c(X, c_c(Y, E))$.

Now for the other direction. Suppose f is in $c(X, c_c(Y, E))$. Define $\nu(f): X \times Y \rightarrow E$ by $\nu(f)(x, y) = f(x)(y)$.

Let K be compact in $X \times Y$. Then K is compact in $X \times Y$. So there exist compact sets L and M in X and Y respectively such that $K \subset L \times M$. I will prove that $\nu(f)|_{L \times M}$ is continuous, which will prove that $\nu(f)|_K$ is continuous. Recall that the relative topology induced on K by $X \times Y$ and by $L \times M$ agree (cf. C4).

Let $(x_0, y_0) \in L \times M$ and U be a neighborhood of

zero in E . Let V be an open neighborhood of zero such that $V + V \subset U$. $f(L)$ is compact in $c_c(Y, E)$, so by Ascoli, $f(L)|M: M \rightarrow E$ is equicontinuous. Hence there exists a neighborhood N_{y_0} of y_0 such that $y \in N_{y_0} \cap M$ implies that $f(\ell)(y) - f(\ell)(y_0) \in V$ for all $\ell \in L$. Now $\{y_0\}$ is compact in Y , so $\Gamma = \{g \in c(Y, E) : g(y_0) \in f(x_0)(y_0) + V\}$ is open in $c_c(Y, E)$. Hence $f^{-1}(\Gamma)$ is open in X and is a neighborhood of x_0 . Also note that $(L \cap f^{-1}(\Gamma)) \times (M \cap N_{y_0})$ is a neighborhood of (x_0, y_0) in $L \times M$.

Let $(x, y) \in (L \cap f^{-1}(\Gamma)) \times (N_{y_0} \cap M)$. Then $f(x)(y_0) - f(x_0)(y_0) \in V$ since $x \in f^{-1}(\Gamma)$ and $f(x)(y) - f(x)(y_0) \in V$ since $y \in N_{y_0} \cap M$ and $x \in L$. So $v(f)(x, y) - v(f)(x_0, y_0) = f(x)(y) - f(x_0)(y_0) \in V + V \subset U$. So $v(f)|L \times M$ is continuous. Hence $v(f)|K$ is continuous. Hence $v(f): X \times Y \rightarrow E$ is continuous, since $X \times Y$ is a k -space. Thus v is a map from $c(X, c_c(Y, E))$ to $c(X \times Y, E)$.

Now to show that v is continuous from $c_c(X, c_c(Y, E))$ to $c_c(X \times Y, E)$. Let K be a compact subset of $X \times Y$. Then K is compact in $X \times Y$ and hence there exist L and M compact in X and Y respectively such that $K \subset L \times M$. Let U be a neighborhood of zero in E . Now $\{f \in c(X, c_c(Y, E)) : f(L) \subset \{g \in c(Y, E) : g(M) \subset U\}\}$ is a basic neighborhood of zero in $c_c(X, c_c(Y, E))$ and is contained in $v^{-1}(\{h \in c(X \times Y, E) : h(K) \subset U\})$. Thus v is continuous.

Obviously $\nu \circ \mu = 1_{c_c(X \boxtimes Y, E)}$ and
 $\mu \circ \nu = 1_{c_c(X, c_c(Y, E))}$. Thus $c_c(X, c_c(Y, E)) \cong c_c(X \boxtimes Y, E)$
 as locally convex topological vector spaces.

The verification that the isomorphisms are "natural" is routine. QED

I will now begin the development of a theorem which says that $C(X, C(Y, E))$ is naturally isomorphic to $C(X \boxtimes Y, E)$. Recall that α is the "p-determination" functor and that $C(X, E) = \alpha(c_c(X, E))$.

16.2 Lemma. If X is a k -space and E is a locally convex Hausdorff space, then $c(X, E) = c(X, \alpha(E))$.

Proof: Since the topology of $\alpha(E)$ is finer than that of E , $c(X, \alpha(E)) \subset c(X, E)$. Suppose $f: X \rightarrow E$ is continuous. Let $K \subset X$ be compact. Then $f(K)$ is compact, hence precompact in E . By 2.1, the uniformity induced on $f(K)$ by $\alpha(E)$ is the same as the uniformity induced on $f(K)$ by E . So $f|_K: K \rightarrow \alpha(E)$ is continuous. But since $X \in \mathcal{J}$, $f: X \rightarrow \alpha(E)$ is continuous. Thus $c(X, E) \subset c(X, \alpha(E))$. QED

16.3 Lemma. Let $X \in \text{ob } \mathcal{J}$ and $E \in \text{ob } \mathcal{C}$, then the topology of $c_c(X, \alpha(E))$ is finer than that of $c_c(X, E)$.

Proof: Trivial from definitions.

16.4 Theorem. The functors $(X, E) \rightarrow C(X, \alpha(E))$ and $(X, E) \rightarrow C(X, E)$ from $\mathcal{J}^{\text{op}} \times \mathcal{C}$ to \mathcal{O} are equal.

Proof: If $f: E \rightarrow F$ is a \mathcal{E} -morphism, then $\alpha(f)$ is the same map as f , the only difference being that the domain and codomain spaces are considered to have different topologies. This implies that the two functors above act the same on morphisms.

Thus there only remains to check that if $X \in \text{ob } \mathcal{J}$ and $E \in \text{ob } \mathcal{C}$, then $\alpha(c_{\mathcal{C}}(X, E)) = \alpha(c_{\mathcal{C}}(X, \alpha(E)))$.

We have already seen that as sets, $c(X, E)$ equals $c(X, \alpha(E))$ and hence $\alpha(c_{\mathcal{C}}(X, E))$ equals $\alpha(c_{\mathcal{C}}(X, \alpha(E)))$. And since the topology of $c_{\mathcal{C}}(X, \alpha(E))$ is finer than that of $c_{\mathcal{C}}(X, E)$, the topology of $\alpha(c_{\mathcal{C}}(X, \alpha(E)))$ is finer than that of $\alpha(c_{\mathcal{C}}(X, E))$. So all that remains is to prove that $\alpha(c_{\mathcal{C}}(X, E))$ is finer than $\alpha(c_{\mathcal{C}}(X, \alpha(E)))$, i.e. to show that the identity map from $\alpha(c_{\mathcal{C}}(X, E))$ to $\alpha(c_{\mathcal{C}}(X, \alpha(E)))$ is continuous. But using the fact that \mathcal{O} is coreflective in \mathcal{E} , this can be accomplished by showing that the identity map from $\alpha(c_{\mathcal{C}}(X, E))$ to $c_{\mathcal{C}}(X, \alpha(E))$ is continuous. Recall $C(X, E) = \alpha(c_{\mathcal{C}}(X, E))$.

To this end, let U be a neighborhood of zero in $\alpha(E)$. Let K be compact in X . Define $d: X \rightarrow \text{hom}(C(X, E), E)$ by $d(x)(f) = f(x)$. Note that $d(K)$ is equicontinuous from $C(X, E)$ to E . Also for all $f \in c(X, E)$, $d(K)(f) = f(K)$ is compact, hence precompact, in E . Thus by Ascoli's theorem $d(K)$ is precompact in $\text{hom}_p(C(X, E), E)$. Hence by 2.17, $d(K)$ is equicontinuous as functions from $C(X, E)$ to $\alpha(E)$. So there exists a neighborhood of zero V in $C(X, E)$ such that $d(K)(V) \subset U$, i.e. $V \subset \{f \in c(X, \alpha(E)) : f(K) \subset U\}$.

So the identity map from $C(X, E)$ to $c_c(X, \alpha(E))$ is continuous. Hence

$$C(X, E) = \alpha(c_c(X, E)) = \alpha(c_c(X, \alpha(E))) = C(X, \alpha(E)). \quad \text{QED}$$

16.5 Theorem. The functors $(X, Y, E) \mapsto C(X, C(Y, E))$ and $(X, Y, E) \mapsto C(X \boxplus Y, E)$ from $\mathcal{J}^{\text{op}} \times \mathcal{J}^{\text{op}} \times \mathcal{C}$ to \mathcal{M} (respectively, from $\mathcal{J}^{\text{op}} \times \mathcal{J}^{\text{op}} \times \mathcal{B}$ to \mathcal{D}) are naturally isomorphic.

Proof: Applying the functor α to theorem 16.1, we find that $\alpha(c_c(X \boxplus Y, E))$ is naturally isomorphic to $\alpha(c_c(X, c_c(Y, E)))$ [cf. A1]. But $\alpha(c_c(X, c_c(Y, E)))$ equals $\alpha(c_c(X, \alpha(c_c(Y, E))))$ by 16.4. Hence $C(X, C(Y, E))$ is naturally isomorphic to $C(X \boxplus Y, E)$ as functors from $\mathcal{J}^{\text{op}} \times \mathcal{J}^{\text{op}} \times \mathcal{C}$ to \mathcal{M} . To get the "respectively" statement, use 10.3. QED

16.6 Theorem. The functors from $\mathcal{J}^{\text{op}} \times \mathcal{J}^{\text{op}} \times \mathcal{B}$ to \mathcal{D} defined by $(X, Y, E) \mapsto \text{Hom}(M(X), \text{Hom}(M(Y), E))$ and $(X, Y, E) \mapsto \text{Hom}(M(X \boxplus Y), E)$ are naturally isomorphic.

Proof: This theorem follows from 16.5 and from 4) of 10.6 by a couple of applications of A1 and A2. QED

16.7 Corollary. The statement of the last theorem remains true if \mathcal{B} is replaced by \mathcal{D} .

Proof: \mathcal{D} is a subcategory of \mathcal{B} . QED

The following lemma will be of value for computations. In this lemma if E is a p -complete locally convex space,

X is a k -space, and $f: X \rightarrow E$ is a continuous function, then $\bar{f}: M(X) \rightarrow E$ will denote the unique continuous linear map such that $\bar{f} \circ \epsilon = f$. Recall that \mathbb{K} denotes the scalars.

16.8 Lemma. Let X and Y be k -spaces. Let $f: X \times Y \rightarrow \mathbb{K}$ be a continuous function. Define continuous maps $f_1: X \rightarrow C(Y)$ and $f_2: Y \rightarrow C(X)$ by $[f_1(x)](y) = f(x, y) = [f_2(y)](x)$. Then $\bar{f}_1(\mu) = \mu \circ f_2$ and $\bar{f}_2(\nu) = \nu \circ f_1$, for all $\mu \in M(X)$ and $\nu \in M(Y)$.

Proof: Let $q: M(X) \rightarrow c_c(Y)$ be defined by $q(\mu) = \mu \circ f_2$. Now $M(X) = C(X)^{\mathbb{P}}$ and f_2 is continuous. Hence $q: M(X) \rightarrow c_c(Y)$, and thus $q: M(X) \rightarrow C(Y)$, is continuous, since $M(X)$ is p -determined. q is also linear.

Let $x \in X$ and $y \in Y$. Then

$[\epsilon(x) \circ f_2](y) = \epsilon(x)[f_2(y)] = [f_2(y)](x) = f(x, y) = [f_1(x)](y)$. So $\epsilon(x) \circ f_2 = f_1(x)$. Hence $q(\epsilon(x)) = f_1(x)$. Hence $q \circ \epsilon = f_1$. So by uniqueness, $\bar{f}_1 = q$. Hence for all $\mu \in M(X)$, $\bar{f}_1(\mu) = q(\mu) = \mu \circ f_2$. That $\bar{f}_2(\nu) = \nu \circ f_1$ for all $\nu \in M(Y)$ is proved

similarly.

QED

Chapter Three

Tensor Products

In this chapter, I define a p -reflexive tensor product which satisfies a universal property with respect to "precompact-precompact" hypocontinuous bilinear forms into p -complete spaces.

Schwartz, in §1 of [41], discusses rather thoroughly various kinds of tensor products and their relationships with one another. My tensor product is very closely related to one of those discussed by Schwartz. It agrees with the projective tensor product of Grothendieck [17] in case the spaces being tensored are both Frechet or both dF ; and it agrees with the greatest cross-norm of Schatten [37] in case the spaces being tensored are Banach. As a matter of fact, if one defines a functor \otimes (which is not the algebraic tensor product) from $\mathcal{D} \times \mathcal{D}$ to \mathcal{D} by $E \otimes F = [\text{Hom}(E, F^p)]^p$, then the functors $(E, F, G) \longmapsto \text{Hom}(E \otimes F, G)$ and $(E, F, G) \longmapsto \text{Hom}(E, \text{Hom}(F, G))$ from $\mathcal{D}^{\text{op}} \times \mathcal{D}^{\text{op}} \times \mathcal{D}$ to \mathcal{D} are naturally isomorphic. One thing to note is that the functors Hom and \otimes take their values in the category \mathcal{D} ; and that the two naturally isomorphic functors, which are described above, also take their values in \mathcal{D} , not in the category of sets. Thus this development is in the spirit of Kelly [24], Linton [28], and Eilenberg and Kelly [14]. Also it should be noted that I have provided a concrete representation for a "completed" type of tensor product.

After a development of the tensor product, various of its properties are studied.

An " ε " product of p -reflexive spaces, analogous to a product defined by Grothendieck (p.89 of [17]), a product defined by Schwartz (§1 of [40]), and the least cross-norm of Schatten [37], is subsequently introduced. Schwartz [40] discusses these notions in a good deal of depth. My " ε " product agrees with that of Schwartz in case the spaces being "epsilon-d" are both Frechet or both dF .

Various amusing relationships between ε and \otimes are proved. Among them, $(E \otimes F)^P \cong E^P \varepsilon F^P$ and $(E \varepsilon F)^P \cong E^P \otimes F^P$.

Finally, the complexification of real p -reflexive spaces and the conjugation of complex p -reflexive spaces are treated.

Section 17 - $\text{Hom}(E \otimes F, G) \cong \text{Hom}(E, \text{Hom}(F, G))$.

17.1 Definitions. If E , F , and G are locally convex spaces and if \mathcal{C} and \mathcal{T} are collections of bounded subsets of E and F respectively, then a collection Q of bilinear forms from $E \times F$ to G is called \mathcal{C} - \mathcal{T} equihypocontinuous provided for all neighborhoods V of zero in G , for all $S \in \mathcal{C}$ and for all $T \in \mathcal{T}$, there exists a neighborhood U_1 of zero in E and a neighborhood U_2 of zero in F such that for all $q \in Q$, $q(S \times U_2) \subset V$ and $q(U_1 \times T) \subset V$.

A single bilinear form $f: E \times F \rightarrow G$ is called \mathcal{C} - \mathcal{T} hypocontinuous provided $\{f\}$ is \mathcal{C} - \mathcal{T} equihypocontinuous. See §4 of Chapter 3 of [4] for a development of these ideas.

We shall now adopt the following conventions for the remainder of this paper.

If a bilinear form or set of bilinear forms is referred to simply as hypocontinuous or equihypocontinuous, then the \mathcal{C} and \mathcal{T} implicitly referred to will be the set of all precompact subsets of the appropriate spaces.

If E , F , and G are locally convex spaces, then $b(E, F; G)$ will denote the set of all \mathcal{C} - \mathcal{T} hypocontinuous bilinear forms from $E \times F$ to G , where \mathcal{C} and \mathcal{T} denote all precompact subsets of E and F respectively.

If E , F , and G are locally convex spaces, then $b_p(E, F; G)$ will denote the locally convex space consisting of $b(E, F; G)$ with the topology of uniform convergence on products of precompact sets.

If E , F , and G are locally convex spaces, then $B(E, F; G)$ will denote $\alpha(b_p(E, F; G))$ where α as usual is the coreflection functor from locally convex spaces to p -determined spaces.

If E and F are locally convex spaces, then $b(E, F)$, $b_p(E, F)$, and $B(E, F)$ will denote $b(E, F; \mathbb{K})$, $b_p(E, F; \mathbb{K})$, and $B(E, F; \mathbb{K})$ respectively, where \mathbb{K} as usual denotes the scalars.

Note: b is a functor from $\mathcal{E}^{op} \times \mathcal{E}^{op} \times \mathcal{E}$ to Sets, b_p is a functor from $\mathcal{E}^{op} \times \mathcal{E}^{op} \times \mathcal{E}$ to \mathcal{C} , and B is a functor from $\mathcal{E}^{op} \times \mathcal{E}^{op} \times \mathcal{E}$ to \mathcal{M} ; provided morphisms are handled in the obvious manner. End of 17.1

The following technical lemmas will be of value before we get into the main body of this section.

17.2 Lemma. Let E and F be p -determined locally convex spaces and let G be a Hausdorff locally convex space. Let \mathcal{C} and \mathcal{Z} denote the collection of all precompact subsets of E and F respectively. Then the following are equivalent:

- A) P is a precompact subset in $\text{hom}_p(E, \text{hom}_p(F, G))$.
- B) P [considered as bilinear forms from $E \times F$ to G] is \mathcal{C} - \mathcal{Z} equihypocontinuous and $P(e, f)$ is precompact in G for all $(e, f) \in E \times F$.

C) P [considered as bilinear forms from $E \times F$ to G] is \mathcal{G} - \mathcal{X} equihypocontinuous and $P(S \times T)$ is precompact in G for all $S \in \mathcal{G}$ and $T \in \mathcal{X}$.

Proof: A) \Rightarrow C). Let P be as in A). By 1.6, P is equicontinuous as functions from E to $\text{hom}_p(F, G)$ and hence P is \mathcal{X} equihypocontinuous [see [4] for definition]. The canonical bilinear form ϕ from $\text{hom}_p(E, \text{hom}_p(F, G)) \times E$ to $\text{hom}_p(F, G)$ is Γ - \mathcal{G} hypocontinuous, where Γ equals all equicontinuous subsets of $\text{hom}(E, \text{hom}_p(F, G))$. So if $S \in \mathcal{G}$, $\phi|_{P \times S}$ is uniformly continuous (cf. prop.5 of chap. 3, §4, of [4]). Thus $\phi(P \times S)$ is precompact in $\text{hom}_p(F, G)$. So again by 1.6, $P(S)$ is equicontinuous from F to G . Hence P is \mathcal{G} equihypocontinuous. If ψ is the canonical bilinear form from

$\text{hom}_p(F, G) \times F$ to G , then ψ is Σ - \mathcal{X} hypocontinuous, where Σ equals all equicontinuous subsets of $\text{hom}(F, G)$. So if $T \in \mathcal{X}$, then $\psi|_{P(S) \times T}$ is uniformly continuous. So $\psi(P(S) \times T) = P(S)(T)$ is precompact in G . Thus for all $S \in \mathcal{G}$ and $T \in \mathcal{X}$, $P(S)(T)$ is precompact in G .

B) \Rightarrow A). Suppose R is a set of \mathcal{G} - \mathcal{X} equihypocontinuous bilinear forms such that for all (e, f) in $E \times F$, $R(e, f)$ is precompact in G . Now R can be regarded as a subset of $\text{hom}(E, \text{hom}_p(F, G))$ since each element of R is $\{\{e\} : e \in E\}$ - \mathcal{X} hypocontinuous. Also R is equicontinuous as functions from E to $\text{hom}_p(F, G)$ because R is \mathcal{X} equihypocontinuous. So by Ascoli's

theorem R will be precompact provided for all $e \in E$, $R(e)$ is precompact in $\text{hom}_p(F,G)$. So let $e \in E$. $R(e)$ is equicontinuous in $\text{hom}(F,G)$ since R is $\{\{e\} : e \in E\}$ equihypocontinuous. Thus again by Ascoli, $R(e)$ will be precompact in $\text{hom}_p(F,G)$ provided $R(e)(f)$ is precompact in G for all $f \in F$. But this is the case since we are assuming B). So $R(e)$ is precompact in $\text{hom}_p(F,G)$. Hence R is precompact as a subset of $\text{hom}_p(E, \text{hom}_p(F,G))$.

C) \Rightarrow B) is obvious. QED

17.3 Lemma. If E and F are p -determined and G is p -complete, then $\text{hom}_p(E, \text{hom}_p(F,G))$ is p -complete.

Proof: Apply 1.23 twice. QED

As a consequence of the last two lemmas we get

17.4 Lemma. If E , F , and G are locally convex spaces with E and F p -determined, then $b_p(E,F;G)$ is \mathcal{C} -isomorphic to $\text{hom}_p(E, \text{hom}_p(F,G))$; and if G is p -complete then so are $b_p(E,F;G)$ and $\text{hom}_p(E, \text{hom}_p(F,G))$.

QED

17.5 Lemma. If E and F are locally convex spaces with E p -determined, then $\text{Hom}(E, \alpha(F)) = \text{Hom}(E, F)$.

Proof: $\text{hom}(E, \alpha(F)) = \text{hom}(E, F)$ as sets because α is a coreflection functor. $\text{hom}_p(E, \alpha(F))$ is finer than $\text{hom}_p(E, F)$ since $\alpha(F)$ is finer than F . So $\alpha(\text{hom}_p(E, \alpha(F)))$ is finer than $\alpha[\text{hom}_p(E, F)]$.

So $\text{Hom}(E, \alpha(F))$ is finer than $\text{Hom}(E, F)$.

Now the canonical linear map from E to $\text{hom}_p(\text{hom}_p(E, F), F)$ is continuous, since precompact subsets of $\text{hom}_p(E, F)$ are equicontinuous. Also the canonical map from $\text{hom}_p(\text{hom}_p(E, F), F)$ to $\text{hom}_p(\text{Hom}(E, F), F)$ is continuous. So the canonical map from E to $\text{hom}_p(\text{Hom}(E, F), F)$ is continuous. Thus the image of a precompact set is precompact and hence equicontinuous as functions from $\text{Hom}(E, F)$ to $\alpha(F)$ (cf. Lemma 2.17).

This says that the map from $\text{Hom}(E, F)$ to $\text{hom}_p(E, \alpha(F))$ is continuous. And hence since $\text{Hom}(E, F)$ is p -determined, the map from $\text{Hom}(E, F)$ to $\alpha(\text{hom}_p(E, \alpha(F))) = \text{Hom}(E, \alpha(F))$ is continuous. Thus $\text{Hom}(E, F)$ is finer than $\text{Hom}(E, \alpha(F))$.

Hence $\text{Hom}(E, F) = \text{Hom}(E, \alpha(F))$. QED

17.6 Corollary. If $E, F,$ and G are locally convex spaces with E and F p -determined, then

$$B(E, F; G) = \text{Hom}(E, \text{Hom}(F, G)).$$

Proof: By 17.4, $b_p(E, F; G) = \text{hom}_p(E, \text{hom}_p(F, G))$. Hence $B(E, F; G) = \alpha(b_p(E, F; G)) = \alpha(\text{hom}_p(E, \text{hom}_p(F, G))) = \text{Hom}(E, \text{hom}_p(F, G))$. But $\text{Hom}(E, \text{hom}_p(F, G)) = \text{Hom}(E, \alpha(\text{hom}_p(F, G)))$ by 17.5. Thus $B(E, F; G) = \text{Hom}(E, \text{Hom}(F, G))$. QED

We are now ready for the main theorem of this section.

17.7 Theorem. If E and F are p -determined spaces

and G is p -complete, then $T \rightarrow T \circ p$ defines a \mathcal{C} -isomorphism from $\text{Hom}(E \otimes F, G)$ to $B(E, F; G) = \text{Hom}(E, \text{Hom}(F, G))$, where $E \otimes F = [B(E, F)]^P$ and $p: E \times F \rightarrow E \otimes F$ is defined by $[p(e, f)](b) = b(e, f)$.

Proof: Let E, F , and G be locally convex spaces with E and F p -determined and G p -complete. Let $d: b_p(E, F; G) \times G^P \rightarrow b_p(E, F)$ be the canonical bilinear form. It is separately continuous and Γ -hypocontinuous where Γ equals all equicontinuous subsets of G' .

Define $e: G^P \rightarrow \text{hom}_p(B(E, F; G), b_p(E, F))$ by $e(\phi)(\eta) = d(\eta, \phi)$.

17.7a Claim: e is continuous.

Proof of claim: Let U be a basic neighborhood of zero in $b_p(E, F)$, i.e. $U = \{b : b(R \times S) \subset \{r : |r| \leq \varepsilon\}\}$ for some $\varepsilon > 0$ and R and S precompact in E and F respectively. Let P be precompact in $B(E, F; G)$. Then P is precompact in $b_p(E, F; G)$. Now $b_p(E, F; G)$ is isomorphic to $\text{hom}_p(E, \text{hom}_p(F, G))$ by 17.4. So 17.2 tells us that $P(R \times S)$ is precompact in G . Let $V = [P(R \times S)]^\circ$. Thus εV is a neighborhood of zero in G^P and $e(\varepsilon V) \subset U$. QED on claim

Now G^P is p -determined since G is p -complete. So d is separately continuous from $B(E, F; G) \times G^P$ to $B(E, F)$. But d is actually Γ -hypocontinuous with these same domain and range spaces by the following reasoning: if $M \in \Gamma$, then $e(M)$ is precompact in $\text{hom}_p(B(E, F; G), b_p(E, F))$ by 17.7a (since equicontinuous

sets are precompact in $G^{\mathbb{P}}$; and thus by 2.17, $e(M)$ is equicontinuous from $B(E,F;G)$ to $\alpha(b_p(E,F)) = B(E,F)$.

Define $c: b(E,F;G) \rightarrow \text{hom}(G^{\mathbb{P}}, B(E,F))$ by $c(\psi)(\phi) = d(\psi, \phi)$. Let $\psi \in b(E,F;G)$ and let M be equicontinuous in G' . Then M is precompact in $G^{\mathbb{P}}$. Thus $c(\psi)(M)$ is thus precompact in $B(E,F)$. So ${}^t[c(\psi)]: B(E,F)^{\mathbb{P}} \rightarrow (G^{\mathbb{P}})'$ is continuous where $(G^{\mathbb{P}})'$ has the topology of uniform convergence on equicontinuous subsets of G' . Let $\delta: G \rightarrow (G^{\mathbb{P}})'$ be the natural isomorphism (recall that G is p -complete). Define $E \otimes F = [B(E,F)]^{\mathbb{P}}$. Notice that $b_p(E,F)$ is p -complete by 17.4, and hence $B(E,F) = \alpha(b_p(E,F))$ is p -reflexive by 2.15. Thus $E \otimes F$ is p -reflexive. Define $p: E \times F \rightarrow E \otimes F$ by $p(e,f)(b) = b(e,f)$ for all $b \in b(E,F)$.

17.7b Claim. p is hypocontinuous.

Proof of claim: Let $S \subset E$ be precompact. Let $\varepsilon > 0$ and P be precompact in $B(E,F)$. P is thus precompact $b_p(E,F)$. By lemmas 17.2 and 17.4, P is equihypocontinuous. So there exists a neighborhood U of zero in F such that $P(S \times U) \subset \{r \in \mathbb{K} : |r| < \varepsilon\}$. So $p(S \times U) \subset \{\phi \in E \otimes F : \phi(P) \subset \{r : |r| < \varepsilon\}\}$. This is one half of the proof that p is hypocontinuous. The other half is entirely analogous. QED on claim

Define $i: b(E,F;G) \rightarrow \text{hom}(E \otimes F, G)$ by $i(\psi) = \delta^{-1} \circ {}^t[c(\psi)]$. Now let $\psi \in b(E,F;G)$.

17.7c Claim. $i(\psi) \circ p = \psi$, or equivalently ${}^t[c(\psi)] \circ p = \delta \circ \psi$.

Proof of claim: Let $\mu \in G'$ and $(e, f) \in E \times F$. Then
 $[{}^t[c(\psi)](p(e, f))](\mu) = [c(\psi)(\mu)](e, f) = d(\psi, \mu)(e, f) =$
 $\mu(\psi(e, f)) = [\delta(\psi(e, f))](\mu)$. So ${}^t[c(\psi)] \circ p = \delta \circ \psi$.

QED on claim

17.7d Theorem. If E and F are p -determined and G is p -complete, then

1) the linear span of the image of p is dense in $E \otimes F$; and

2) if $b: E \otimes F \rightarrow G$ is a hypocontinuous bilinear form, then there exists a unique continuous linear map $\bar{b}: E \otimes F \rightarrow G$ such that $\bar{b} \circ p = b$.

Proof of 17.7d: Existence of 2) is proved in 17.7c.

Suppose S and T are continuous linear maps from $E \otimes F$ to G such that $S \circ p = T \circ p$. Let R equal the closed linear span of the image of p . Since G is Hausdorff, we have $R \subset (S - T)^{-1}(\{0\})$. So if I can prove that $R = E \otimes F$, then I will have proved both 1) and the uniqueness part of 2).

Let μ be an element of $(E \otimes F)'$ such that $\mu(r) = 0$ for all $r \in R$. Since $B(E, F)$ is p -complete, $(E \otimes F)' = b(E, F)$. So there exists a $b \in b(E, F)$ such that $\mu(x) = x(b)$ for all $x \in E \otimes F$. Hence $0 = \mu(p(e, f)) = p(e, f)(b) = b(e, f)$ for all $(e, f) \in E \times F$. Hence $b = 0$. So $\mu = 0$. Hence $R = E \otimes F$, by the Hahn-Banach theorem. QED on 17.7d

Now I would like to show that i is a continuous linear map from $B(E, F; G)$ to $\text{hom}_p(E \otimes F, G)$. To this end, let P be a precompact subset of $E \otimes F$ and let

U be a closed, convex, balanced neighborhood of zero in G . Since $B(E, F)$ is p -determined, P is equicontinuous as functions from $B(E, F)$ to \mathbb{K} . So there exists a neighborhood V of zero in $B(E, F)$ such that $|b(V)| \subset \{k \in \mathbb{K} : |t| \leq 1\}$ for all $b \in P$. Now U° is an equicontinuous subset of G' . So by the Γ -hypocontinuity of d , there exists a neighborhood W of zero in $B(E, F; G)$ such that $d(W \times U^\circ) \subset V$. I claim $i(W) \subset \{T \in \text{hom}(E \otimes F, G) : T(P) \subset U\}$.

Let $w \in W$ and $b \in P$. Is $i(w)(b) \in U$? Let $\phi \in U^\circ \subset G'$. Then $|b(d(w, \phi))| \leq 1$. But $b(d(w, \phi)) = b(c(w)(\phi)) = {}^t[c(w)](b)(\phi)$. So ${}^t[c(w)](b)$ is in $U^{\circ\flat}$. So $\delta^{-1} \circ {}^t[c(w)](b) \in \delta^{-1}(U^{\circ\flat}) \subset U^{\circ\flat} \subset U$. Thus $i(w)(b) \in U$.

Thus $i: B(E, F; G) \rightarrow \text{hom}_p(E \otimes F, G)$ is continuous. Hence $i: B(E, F; G) \rightarrow \text{Hom}(E \otimes F, G)$ is continuous, since $B(E, F; G)$ is p -determined.

Define $j: \text{hom}(E \otimes F, G) \rightarrow b(E, F; G)$ by $j(T) = T \circ p$. [Recall that p is hypocontinuous.] Note that $j \circ i = 1_{b(E, F; G)}$ and $i \circ j = 1_{\text{hom}(E \otimes F, G)}$ due to 17.7d. Since p is hypocontinuous, if S and T are precompact subsets of E and F respectively, then $p|_{S \times T}$ is uniformly continuous, so $p(S \times T)$ is precompact. Thus j is continuous from $\text{hom}_p(E \otimes F, G)$ to $b_p(E, F; G)$. So $j: \alpha(\text{hom}_p(E \otimes F, G)) \rightarrow \alpha(b_p(E, F; G))$ is continuous.

Hence both $i: B(E, F; G) \rightarrow \text{Hom}(E \otimes F, G)$ and $j: \text{Hom}(E \otimes F, G) \rightarrow B(E, F; G)$ are \mathcal{C} -isomorphisms. Note that both $B(E, F; G)$ and $\text{Hom}(E \otimes F, G)$ are p -reflexive.

We have thus shown that if G is p -complete and E and F are p -determined, then $\text{Hom}(E, \text{Hom}(F, G)) = B(E, F; G) \cong \text{Hom}(E \otimes F, G)$, the equality is from 17.6 and the isomorphism follows from above. QED

17.8 Definition. If E and F are p -determined spaces, then define a p -reflexive space $E \otimes F = B(E, F)^P$ and define a hypocontinuous bilinear form $p: E \times F \rightarrow E \otimes F$ by $p(e, f)(b) = b(e, f)$. If $f \in \text{Mor}_{\mathcal{O}}(A, A_1)$ and $g \in \text{Mor}_{\mathcal{O}}(B, B_1)$, then define $f \otimes g$ equal to the unique continuous linear map from $A \otimes B$ to $A_1 \otimes B_1$ such that

$$\begin{array}{ccc} A \times B & \xrightarrow{f \times g} & A_1 \times B_1 \\ p \downarrow & & \downarrow p \\ A \otimes B & \dashrightarrow & A_1 \otimes B_1 \end{array} \quad \text{commutes.}$$

17.9 Remark. Everything needed to justify the above definition is contained in the proof of 17.7.

17.10 Proposition. \otimes is a functor from $\mathcal{O} \times \mathcal{O}$ to \mathcal{D} .

Proof: If $A, B \in \mathcal{O}$, then the identity map is a map making

$$\begin{array}{ccc} & A \times B & \\ p \swarrow & & \searrow p \\ A \otimes B & \dashrightarrow & A \otimes B \end{array} \quad \text{commute.}$$

Hence $l_A \otimes l_B = l_{A \otimes B}$.

Suppose $(f, h) \in \text{Mor}_{\mathcal{O} \times \mathcal{O}}((A, B), (A_1, B_1))$ and $(g, j) \in \text{Mor}_{\mathcal{O} \times \mathcal{O}}((A_1, B_1), (A_2, B_2))$. Then consider

$$\begin{array}{ccccc}
 A \times B & \xrightarrow{f \times h} & A_1 \times B_1 & \xrightarrow{g \times j} & A_2 \times B_2 \\
 \downarrow p & & \downarrow p & & \downarrow p \\
 A \otimes B & \xrightarrow{f \otimes h} & A_1 \otimes B_1 & \xrightarrow{g \otimes j} & A_2 \otimes B_2
 \end{array}$$

Both of the small diagrams commute by the definition of $f \otimes h$ and $g \otimes j$. Hence the large diagram consisting of the outside arrows commutes. But the commuting of the large diagram tells us that $(g \circ f) \otimes (j \circ h) = (g \otimes j) \circ (f \otimes h)$. Hence \otimes is a functor. QED

17.11 Theorem. The functors $(E, F, G) \mapsto \text{Hom}(E \otimes F, G)$ and $(E, F, G) \mapsto \text{Hom}(E, \text{Hom}(F, G))$ from $\mathcal{O}^{\text{op}} \times \mathcal{O}^{\text{op}} \times \mathcal{C}$ to \mathcal{C} are naturally isomorphic, the natural isomorphism being defined by $T \mapsto T \circ p$.

Proof: For all $(E, F, G) \in \mathcal{O}^{\text{op}} \times \mathcal{O}^{\text{op}} \times \mathcal{C}$ let $j: \text{Hom}(E \otimes F, G) \rightarrow \text{Hom}(E, \text{Hom}(F, G))$ denote the isomorphism of 17.7, i.e. $j(T) = T \circ p$. Suppose A, B, A', B' are in $\text{ob } \mathcal{O}$ and C and C' are in $\text{ob } \mathcal{C}$. Suppose $f: A' \rightarrow A$, $g: B' \rightarrow B$, and $h: C \rightarrow C'$. Does the diagram

$$\begin{array}{ccc}
 \text{Hom}(A \otimes B, C) & \xrightarrow{\text{Hom}(f \otimes g, h)} & \text{Hom}(A' \otimes B', C') \\
 \downarrow j & & \downarrow j \\
 \text{Hom}(A, \text{Hom}(B, C)) & \xrightarrow{\text{Hom}(f, \text{Hom}(g, h))} & \text{Hom}(A', \text{Hom}(B', C'))
 \end{array}$$

commute? Suppose $T \in \text{Hom}(A \otimes B, C)$. Then

$$\begin{aligned}
 [j \circ \text{Hom}(f \otimes g, h)](T) &= j[h \circ T \circ (f \otimes g)] = h \circ T \circ (f \otimes g) \circ p \quad \text{and} \\
 [\text{Hom}(f, \text{Hom}(g, h)) \circ j](T) &= h \circ T \circ p \circ (f \times g). \quad \text{But by 17.8,}
 \end{aligned}$$

$p \circ (f \times g) = (f \otimes g) \circ p$. So

$[\text{Hom}(f, \text{Hom}(g, h)) \circ j](T) = [j \circ \text{Hom}(f \otimes g, h)](T)$. But this is true for all such T . So the above diagram commutes. So the j 's form a natural isomorphism. QED

17.12 Convention. If E and F are p -determined spaces and $e \in E$ and $f \in F$, then $e \otimes f$ will frequently be used to denote $p(e, f)$ where $p: E \times F \rightarrow E \otimes F$ is as in 17.8. Elements of $E \otimes F$ of the form $e \otimes f$ where $e \in E$ and $f \in F$ will be called elementary tensors. In the same spirit \otimes will sometimes be used to denote the hypocontinuous bilinear map p from $E \times F$ to $E \otimes F$. Thus if $A \subset E$ and $B \subset F$, $A \otimes B$ will denote $\{p(a, b) : a \in A \text{ \& } b \in B\} = \{a \otimes b : a \in A \text{ \& } b \in B\}$.

Section 18 - Various properties of the tensor product.

18.0 Notation. If E and F are locally convex spaces, and $A \subset E$ and $B \subset F$, then define $N(E, F; A, B)$ to be equal to $\{T \in \text{hom}(E, F) : T(A) \subset B\}$.

18.1 Proposition. Let E and F be p -determined spaces. Let A and B be subsets of E and F respectively, whose linear spans are dense in E and F respectively. Then the linear span of $\{a \otimes b : a \in A \text{ and } b \in B\}$ is dense in $E \otimes F$.

Proof: Let C equal the closure of the linear span of $\{a \otimes b : a \in A \text{ and } b \in B\}$. Let $\phi: E \otimes F \rightarrow \mathbb{K}$ be a linear functional such that $\phi(c) = 0$ for all $c \in C$. Let j be the natural isomorphism from $\text{Hom}(E \otimes F, \mathbb{K})$ to $\text{Hom}(E, \text{Hom}(F, \mathbb{K}))$. Let $a \in A$. Is $[j(\phi)](a) = 0$? Well, let's check. For all $b \in B$, $[[j(\phi)](a)](b) = 0$; because $\phi(a \otimes b) = 0$ for all $b \in B$. So because \mathbb{K} is Hausdorff and the linear span of B is dense in F , we see that $[j(\phi)](a) = 0$. But this is true for all $a \in A$. Hence $j(\phi) = 0$, since $\text{Hom}(F, \mathbb{K})$ is Hausdorff. But then $\phi = 0$. Hence $C = E \otimes F$ by the Hahn-Banach theorem. QED

18.2 Theorem. Let E and F be p -determined spaces. Let $E \otimes F = B(E, F)^P$ and $p: E \times F \rightarrow E \otimes F$ be the evaluation map [i.e. $p(e, f)(b) = b(e, f)$]. Let \mathcal{G} and \mathcal{F}

be all precompact subsets of E and F respectively. Then $E \otimes F$ is p -reflexive and p is \mathcal{C} - \mathcal{X} hypocontinuous. Moreover if G is a p -complete space and $b: E \times F \rightarrow G$ is \mathcal{C} - \mathcal{X} hypocontinuous, then there exists a unique continuous linear map $\bar{b}: E \otimes F \rightarrow G$ such that

$$\begin{array}{ccc}
 E \times F & \xrightarrow{b} & G \\
 p \downarrow & \nearrow & \\
 E \otimes F & & \bar{b}
 \end{array}
 \quad \text{commutes.}$$

Moreover the linear span of the image of p is dense in $E \otimes F$.

Proof: This is a restatement of 17.7d.

18.3 Corollary. There exist natural isomorphisms

$$l = l_A: K \otimes A \rightarrow A \quad \text{for all } A \in \text{ob}\mathcal{D};$$

$$r = r_A: A \otimes K \rightarrow A \quad \text{for all } A \in \text{ob}\mathcal{D};$$

$$c = c_{AB}: A \otimes B \rightarrow B \otimes A \quad \text{for all } A \text{ and } B \in \text{ob}\mathcal{M};$$

and $a = a_{ABC}: A \otimes (B \otimes C) \rightarrow (A \otimes B) \otimes C$ for all $A, B,$ and $C \in \text{ob}\mathcal{M}$.

Moreover these natural isomorphisms are defined by

$$k \otimes a \mapsto ka$$

$$a \otimes k \mapsto ka$$

$$a \otimes b \mapsto b \otimes a$$

and $a \otimes (b \otimes c) \mapsto (a \otimes b) \otimes c$ respectively.

Proof: Case r . Let $A \in \mathcal{D}$. Because the bilinear map $p: A \times K \rightarrow A \otimes K$ is separately continuous, the map $\phi: A \rightarrow A \otimes K$ defined by $a \mapsto a \otimes 1$ is continuous.

Define another bilinear map $j: A \times \mathbb{K} \longrightarrow A$ by $j(a,k) = ka$. j is continuous since A is a topological vector space. Hence j is hypocontinuous. So by the last theorem, there exists a unique continuous linear map $r: A \otimes \mathbb{K} \longrightarrow A$ such that $r \circ p = j$. Consider $\phi \circ r$ and $r \circ \phi$. Now $(\phi \circ r) \circ p = p$ since $\phi[(r \circ p)(a,k)] = \phi(j(a,k)) = \phi(ka) = p(ka,1) = p(a,k)$. So $\phi \circ r = 1_{A \otimes \mathbb{K}}$ by 18.2. On the other hand, $r \circ \phi = 1_A$ since if $a \in A$, $r(\phi(a)) = r(p(a,1)) = j(a,1) = 1a = a$. So r is an isomorphism. Now to show that the r 's are natural.

Suppose $f: A_1 \longrightarrow A_2$. Does

$$\begin{array}{ccc}
 A_1 \otimes \mathbb{K} & \xrightarrow{f \otimes 1_{\mathbb{K}}} & A_2 \otimes \mathbb{K} \\
 \downarrow r & & \downarrow r \\
 A_1 & \xrightarrow{f} & A_2
 \end{array}
 \quad \text{commute?}$$

By 18.1, it suffices to check the commutativity on elementary tensors. Let $a \in A_1$ and $k \in \mathbb{K}$. Then $[r \circ (f \otimes 1_{\mathbb{K}})](a \otimes k) = r(f(a) \otimes k) = kf(a) = f(ka) = f(r(a \otimes k)) = (f \circ r)(a \otimes k)$. So the diagram commutes. Hence r is natural.

Case ℓ . The proof is exactly the same as for r .

Case c . Let A and $B \in \mathcal{O}$. Since $p: B \times A \longrightarrow B \otimes A$ is hypocontinuous, the map from $A \times B$ to $B \otimes A$ defined by $(a,b) \longmapsto p(b,a)$ is hypocontinuous. Hence by 18.2, there exists a unique continuous linear map c from $A \otimes B$ to $B \otimes A$ such that $c(a \otimes b) = b \otimes a$ for all

$(a,b) \in A \times B$. By the same argument there exists a continuous linear map j from $B \otimes A$ to $A \otimes B$ such that $j(b \otimes a) = a \otimes b$ for all $(a,b) \in A \times B$. Hence by 18.1, $j \circ c = 1_{A \otimes B}$ and $c \circ j = 1_{B \otimes A}$. Hence c is an isomorphism. Next to check that c is natural.

Suppose $f: A \rightarrow C$ and $g: B \rightarrow D$. Does

$$\begin{array}{ccc}
 A \otimes B & \xrightarrow{f \otimes g} & C \otimes D \\
 \downarrow c & & \downarrow c \\
 B \otimes A & \xrightarrow{g \otimes f} & D \otimes C
 \end{array}
 \quad \text{commute?}$$

It suffices to check it for elementary tensors, so let $(a,b) \in A \times B$. Then

$$\begin{aligned}
 [c \circ (f \otimes g)](a \otimes b) &= c(f(a) \otimes g(b)) = g(b) \otimes f(a) \quad \text{and} \\
 [(g \otimes f) \circ c](a \otimes b) &= (g \otimes f)(b \otimes a) = g(b) \otimes f(a).
 \end{aligned}$$

So the diagram commutes. Hence c is natural.

Case a. By 17.11 and A1, $\text{Hom}((A \otimes B) \otimes C, D) \cong \text{Hom}(A \otimes B, \text{Hom}(C, D)) \cong \text{Hom}(A, \text{Hom}(B, \text{Hom}(C, D))) \cong \text{Hom}(A, \text{Hom}(B \otimes C, D)) \cong \text{Hom}(A \otimes (B \otimes C), D)$. Thus A5 implies that $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$; and furthermore the isomorphism can be obtained by chasing $1_{(A \otimes B) \otimes C}$ from left to right through the above isomorphisms for the case $D = (A \otimes B) \otimes C$. In this manner it is easy to see that elements of the form $a \otimes (b \otimes c)$ behave under the isomorphism as I say they do. QED on 18.3

18.4 Theorem. Let $B \in \mathcal{O}$. If functors $R': \mathcal{O} \rightarrow \mathcal{L}$ and $S': \mathcal{L} \rightarrow \mathcal{O}$ are defined by $R'(A) = A \otimes B$ and

$S'(C) = \text{Hom}(B, C)$, then R' is an adjoint of S' .

18.5 Theorem. Let $B \in \mathcal{M}$. If functors $R: \mathcal{D} \rightarrow \mathcal{D}$ and $S: \mathcal{D} \rightarrow \mathcal{D}$ are defined by $R(A) = A \otimes B$ and $S(C) = \text{Hom}(B, C)$, then R is an adjoint of S .

Proof of 18.4 and 18.5: 17.11.

QED

18.6 Corollary. R' sends \mathcal{M} -colimits to \mathcal{D} -colimits, S' sends \mathcal{D} -limits to \mathcal{M} -limits, R sends \mathcal{D} -colimits to \mathcal{D} -colimits, and S sends \mathcal{D} -limits to \mathcal{D} -limits.

Proof: Functors which possess adjoints preserve limits and functors which possess coadjoints preserve colimits (cf. 16.4.6 of [39]).

QED

Recall that locally convex direct sums, inductive limits, quotient maps, and maps with dense images are all examples of colimits; and products, projective limits, isomorphisms onto closed subspaces, and injective maps are all examples of limits. Thus corollary 18.6 contains a good deal of information.

18.7 Corollary. Let $B \in \text{ob } \mathcal{D}$. If functors $R: \mathcal{D} \rightarrow \mathcal{D}$ and $S: \mathcal{D}^{\text{op}} \rightarrow \mathcal{D}$ are defined by $R(D) = B \otimes D$ and $S(D) = \text{Hom}(D, B)$, then R has a coadjoint and S has an adjoint.

Proof: We know that $A \otimes B \cong B \otimes A$ naturally. So the functor $D \mapsto D \otimes B$ is naturally isomorphic to R . But the former has an coadjoint and hence by A4 so does R .

Recall from 3.2 the functor $R: \mathcal{D} \rightarrow \mathcal{D}^{\text{OP}}$. By 18.5, the functor $D \mapsto \text{Hom}(R(B), D)$ from \mathcal{D} to \mathcal{D} has an adjoint. Also by 3.9, $R^{\text{OP}}: \mathcal{D}^{\text{OP}} \rightarrow \mathcal{D}$ has an adjoint. So by A3, the composite $D \mapsto \text{Hom}(R(B), R^{\text{OP}}(D))$ from \mathcal{D}^{OP} to \mathcal{D} has an adjoint. This last functor is precisely $D \mapsto \text{Hom}^{\text{OP}}(R^{\text{OP}}(D), R(B))$ (cf. 3.1). But by b) of 3.8 and by A1, this is naturally isomorphic to S . Thus by A4, S has an adjoint. QED

18.8 Corollary. R preserves colimits and S preserves limits. QED

Recall that limits in \mathcal{D}^{OP} are colimits in \mathcal{D} . So in particular, we have that $\text{Hom}(\oplus E_{\alpha}, B) \cong \prod \text{Hom}(E_{\alpha}, B)$.

18.9 Theorem. There exists a natural transformation ϕ from the functor $(A, B, C, D) \mapsto \text{Hom}(A, B) \otimes \text{Hom}(C, D)$ to the functor $(A, B, C, D) \mapsto \text{Hom}(A \otimes C, B \otimes D)$ such that $\phi(S \otimes T) = S \otimes T$. These two functors are being regarded as going from $\mathcal{A}^{\text{OP}} \times \mathcal{A} \times \mathcal{A}^{\text{OP}} \times \mathcal{A}$ to \mathcal{D} .

Note: $S \otimes T$ is used in two different senses above. The first use refers to an elementary tensor in $\text{Hom}(A, B) \otimes \text{Hom}(C, D)$ and the second refers to the action of the functor \otimes on the morphism (S, T) .

Proof: Let $A, B, C, D \in \text{ob } \mathcal{A}$. Let

$$\pi_1: \text{hom}(\text{Hom}(A, B), \text{Hom}(A, B)) \longrightarrow \text{hom}(\text{Hom}(A, B) \otimes A, B) ,$$

$$\pi_2: \text{hom}(\text{Hom}(C, D), \text{Hom}(C, D)) \longrightarrow \text{hom}(\text{Hom}(C, D) \otimes C, D) , \text{ and}$$

$$\pi_3: \text{hom}(\text{Hom}(A, B) \otimes \text{Hom}(C, D), \text{Hom}(A \otimes C, B \otimes D)) \longrightarrow$$

$$\text{hom}([\text{Hom}(A, B) \otimes \text{Hom}(C, D)] \otimes [A \otimes C], B \otimes D) \text{ denote the}$$

canonical natural isomorphisms (cf. 17.11). Let $j = \pi_1(1_{\text{Hom}(A,B)})$ and $k = \pi_2(1_{\text{Hom}(C,D)})$. Note that $j(S \otimes a) = S(a)$ and $k(T \otimes b) = T(b)$. Then $j \otimes k \in \text{hom}([\text{Hom}(A,B) \otimes A] \otimes [\text{Hom}(C,D) \otimes C], B \otimes D)$. Let $m: [\text{Hom}(A,B) \otimes \text{Hom}(C,D)] \otimes [A \otimes C] \rightarrow [\text{Hom}(A,B) \otimes A] \otimes [\text{Hom}(C,D) \otimes C]$ be the "middle-four exchange", i.e. the unique continuous linear map such that $(S \otimes T) \otimes (a \otimes c)$ is sent to $(S \otimes a) \otimes (T \otimes c)$ [use c and a of 18.3 for existence and 18.1 for uniqueness]. [See 23.5 and what follows for a discussion of m].

Define $\phi \in \text{hom}(\text{Hom}(A,B) \otimes \text{Hom}(C,D), \text{Hom}(A \otimes C, B \otimes D))$ by $\phi = \pi_3^{-1}[(j \otimes k) \circ m]$. Let $R \in \text{Hom}(A,B)$, $S \in \text{Hom}(C,D)$, $a \in A$, and $c \in C$. Then

$$\begin{aligned} [\phi(R \otimes S)](a \otimes c) &= [(j \otimes k) \circ m][\{(R \otimes S) \otimes (a \otimes c)\}] = \\ &= (j \otimes k)[\{(R \otimes a) \otimes (S \otimes c)\}] = j(R \otimes a) \otimes k(S \otimes c) = R(a) \otimes S(c). \end{aligned}$$

But $R \otimes S: A \otimes C \rightarrow B \otimes D$ by definition is the unique continuous linear map with this property. Thus $\phi(R \otimes S) = R \otimes S$.

All that remains is to show that ϕ is natural.

Suppose we have \mathcal{O} -morphisms, $f: A' \rightarrow A$, $g: B \rightarrow B'$, $h: C' \rightarrow C$, and $\ell: D \rightarrow D'$. Then does

$$\begin{array}{ccc} \text{Hom}(A,B) \otimes \text{Hom}(C,D) & \xrightarrow{\text{Hom}(f,g) \otimes \text{Hom}(h,\ell)} & \text{Hom}(A',B') \otimes \text{Hom}(C',D') \\ \downarrow \phi & & \downarrow \phi \\ \text{Hom}(A \otimes C, B \otimes D) & \xrightarrow{\text{Hom}(f \otimes h, g \otimes \ell)} & \text{Hom}(A' \otimes C', B' \otimes D') \end{array}$$

commute? It suffices to check the diagram on elementary tensors. So suppose $T \in \text{Hom}(C,D)$ and $S \in \text{Hom}(A,B)$. Then $S \otimes T \in \text{Hom}(A,B) \otimes \text{Hom}(C,D)$. Also

$[\text{Hom}(f, g) \otimes \text{Hom}(h, \ell)](S \otimes T) = (g \circ S \circ f) \otimes (\ell \circ T \circ h)$ and
 $\phi[(g \circ S \circ f) \otimes (\ell \circ T \circ h)] = (g \circ S \circ f) \otimes (\ell \circ T \circ h)$. But
 $\text{Hom}(f \otimes h, g \otimes \ell)[\phi(S \otimes T)] = (g \otimes \ell) \circ [\phi(S \otimes T)] \circ (f \otimes h) =$
 $(g \otimes \ell) \circ (S \otimes T) \circ (f \otimes h) = (g \circ S \circ f) \otimes (\ell \circ T \circ h)$, the last equality
 being due to \otimes being a functor. So these two quantities
 are equal and thus the diagram commutes. QED

18.10 Proposition. The functor from \mathcal{D} to \mathcal{D} defined
 by $E \mapsto \text{Hom}(\mathbb{K}, E)$ is naturally isomorphic to the
 identity functor on \mathcal{D} .

Proof: If $E \in \text{ob } \mathcal{D}$, define $\phi_E: \text{Hom}(\mathbb{K}, E) \rightarrow E$ by
 $\phi_E(f) = f(1)$. Suppose $f_\alpha \rightarrow 0$ in $\text{Hom}(\mathbb{K}, E)$, then
 $f_\alpha \rightarrow 0$ uniformly on precompact sets of \mathbb{K} . So
 $f_\alpha(1) \rightarrow 0$. Thus ϕ_E is continuous. Suppose $e \in E$.
 Define a continuous function $f: \mathbb{K} \rightarrow E$ by $f(\alpha) = \alpha e$.
 Note that $\phi_E(f) = f(1) = e$. So ϕ_E is surjective.
 Linearity and the fact that $\{1\}$ spans \mathbb{K} guarantees
 that ϕ_E is injective.

Suppose $f_\alpha \in \text{Hom}(\mathbb{K}, E)$ and $\phi_E(f_\alpha) = f_\alpha(1) \rightarrow 0$.

Suppose C is a precompact subset of \mathbb{K} and U is a

neighborhood of zero in E . Now the canonical bilinear form from $K \times E \rightarrow E$ is continuous, so it is hypocontinuous. So there exists a neighborhood V of zero in E such that $CV \subset U$. Choose β such that $\alpha \geq \beta$ implies $f_\alpha(1) \in V$. Then $\alpha \geq \beta$ implies $f_\alpha(C) \subset U$, since if $\alpha \geq \beta$ and $c \in C$, $f_\alpha(c) = cf_\alpha(1) \in CV \subset U$. So $f_\alpha \rightarrow 0$ in $\text{hom}_p(K, E)$. Thus $(\phi_E)^{-1}: E \rightarrow \text{hom}_p(K, E)$ is continuous. But since E is p -determined, ϕ_E^{-1} is continuous from E to $\text{Hom}(K, E)$. Hence ϕ_E is an isomorphism.

ϕ is obviously natural, because if $f: E \rightarrow F$ and $T \in \text{Hom}(K, E)$, $(f \circ T)(1) = f(T(1))$. QED

18.11 Corollary.

1) There exists a natural transformation from $E^p \otimes F^p \rightarrow (E \otimes F)^p$ for all E and $F \in \text{ob} \mathcal{D}$. [Regard the functors as from $\mathcal{D}^{\text{op}} \times \mathcal{D}^{\text{op}}$ to \mathcal{D} .]

2) There exists a natural transformation from $E^p \otimes F$ to $\text{Hom}(E, F)$ for all E and $F \in \text{ob} \mathcal{D}$. [Regard the functors as from $\mathcal{D}^{\text{op}} \times \mathcal{D} \rightarrow \mathcal{D}$.]

Proof: By 18.3, there is a natural isomorphism from $(E \otimes K, K \otimes F)$ to (E, F) . So by A1, the map induced by that one from $\text{Hom}(E \otimes K, K \otimes F)$ to $\text{Hom}(E, F)$ is a natural isomorphism. By 18.9, the map from $\text{Hom}(E, K) \otimes \text{Hom}(K, F)$ to $\text{Hom}(E \otimes K, K \otimes F)$ is a natural transformation. By 18.10, $F \mapsto \text{Hom}(K, F)$ is a natural isomorphism. So by composing things, the map from $\text{Hom}(E, K) \otimes F \rightarrow \text{Hom}(E, F)$ is a natural transformation. But $E^p = \text{Hom}(E, K)$. So 2) is

done.

As for 1), note that by 18.9 there exists a natural transformation from $\text{Hom}(E, \mathbb{K}) \otimes \text{Hom}(F, \mathbb{K})$ to $\text{Hom}(E \otimes F, \mathbb{K} \otimes \mathbb{K}) = \text{Hom}(E \otimes F, \mathbb{K})$. But $E^{\mathbb{P}} = \text{Hom}(E, \mathbb{K})$, etc.

QED

18.12 Theorem.

1) The functors $(E, F) \mapsto (E \otimes F)^{\mathbb{P}}$ and $(E, F) \mapsto \text{Hom}(E, F^{\mathbb{P}})$ from $\mathcal{B}^{\text{op}} \times \mathcal{B}^{\text{op}}$ to \mathcal{B} are naturally isomorphic.

2) The functors $(E, F) \mapsto (\text{Hom}(E, F))^{\mathbb{P}}$ and $(E, F) \mapsto E \otimes F^{\mathbb{P}}$ from $\mathcal{B}^{\text{op}} \times \mathcal{B}$ to \mathcal{B}^{op} are naturally isomorphic.

Proof: 1) $\text{Hom}(E \otimes F, \mathbb{K}) \cong \text{Hom}(E, \text{Hom}(F, \mathbb{K})) = \text{Hom}(E, F^{\mathbb{P}})$. 2) $\text{Hom}(E, F) \cong \text{Hom}(E, F^{\mathbb{P}\mathbb{P}}) \cong \text{Hom}(E \otimes F^{\mathbb{P}}, \mathbb{K}) = (E \otimes F^{\mathbb{P}})^{\mathbb{P}}$. So $(E \otimes F^{\mathbb{P}})^{\mathbb{P}\mathbb{P}} \cong (\text{Hom}(E, F))^{\mathbb{P}}$. But $(E \otimes F^{\mathbb{P}})^{\mathbb{P}} \cong (E \otimes F^{\mathbb{P}})^{\mathbb{P}\mathbb{P}}$. So $E \otimes F^{\mathbb{P}} \cong (\text{Hom}(E, F))^{\mathbb{P}}$. QED

Section 19 - The ε product.

I define a functor ε from $\mathcal{D} \times \mathcal{D}$ to \mathcal{D} such that $(E \varepsilon F)^{\mathcal{P}} \cong E^{\mathcal{P}} \otimes F^{\mathcal{P}}$ and $(E \otimes F)^{\mathcal{P}} \cong E^{\mathcal{P}} \varepsilon F^{\mathcal{P}}$.

Laurent Schwartz in exposes 7 and 8 of [42] and §1 of [40] discusses a similar functor. The relationship between his and mine is that mine is the coreflection in the category of p -determined spaces of his.

The algebraic tensor product with the ε -topology which is discussed by many authors (and was introduced by Grothendieck [p.89 of [17]]) can be regarded as a subspace of the Schwartz ε -product with the relative topology. And a locally convex Hausdorff space L satisfies the approximation property (cf. prop. 11 of §1 of [40]) iff for all locally convex Hausdorff spaces M , the algebraic tensor product of L and M is dense in $L (\varepsilon_{\text{Schwartz}}) M$.

19.1 Definition. If E and F are p -reflexive, then define $E \varepsilon F = \text{Hom}(E^{\mathcal{P}}, F)$.

19.2 Lemma. $(E, F) \mapsto E \varepsilon F$ is a functor from $\mathcal{D} \times \mathcal{D}$ to \mathcal{D} .

Proof: Hom is a functor from $\mathcal{D}^{\text{op}} \times \mathcal{D}$ to \mathcal{D} and $E \mapsto E^{\mathcal{P}}$ is a functor from \mathcal{D} to \mathcal{D}^{op} . So composing functors does the trick. QED

19.3 Theorem. In what follows let \cong mean "there exists a natural isomorphism of functors" and let \longrightarrow mean "there exists a natural transformation of functors".

Then

- a) $(E \varepsilon F) \varepsilon G \cong E \varepsilon (F \varepsilon G)$;
 - b) $E \varepsilon F \cong F \varepsilon E$;
 - c) $K \varepsilon E \cong E \cong E \varepsilon K$;
 - d) if F is fixed, then both $E \longmapsto E \varepsilon F$ and $E \longmapsto F \varepsilon E$ have adjoints; hence both preserve limits;
 - e) $(E \varepsilon F)^P \cong E^P \otimes F^P$;
 - f) $(E \otimes F)^P \cong E^P \varepsilon F^P$;
 - g) $(E \varepsilon F) \otimes (G \varepsilon H) \longrightarrow (E \varepsilon G) \varepsilon (F \otimes H)$;
 - h) $(E \otimes G) \otimes (F \varepsilon H) \longrightarrow (E \otimes F) \varepsilon (G \otimes H)$;
- and i) $E \otimes F \longrightarrow E \varepsilon F$.

Proof: e) $(E \varepsilon F)^P = [\text{Hom}(E^P, F)]^P \cong E^P \otimes F^P$ by 18.12.

a) $(E \varepsilon F) \varepsilon G = \text{Hom}((E \varepsilon F)^P, G) \cong \text{Hom}(E^P \otimes F^P, G) \cong \text{Hom}(E^P, \text{Hom}(F^P, G)) = E \varepsilon (F \varepsilon G)$, by using e) among other things.

b) $E \varepsilon F = \text{Hom}(E^P, F) \cong \text{Hom}(F^P, E^{PP}) \cong \text{Hom}(F^P, E) = F \varepsilon E$.

c) $E \varepsilon K = \text{Hom}(E^P, K) = E^{PP} \cong E$. The other isomorphism follows from b).

d) $\text{Hom}(G, F \varepsilon E) = \text{Hom}(G, \text{Hom}(F^P, E)) \cong \text{Hom}(G \otimes F^P, E)$.

So $G \longmapsto G \otimes F^P$ is an adjoint of $E \longmapsto F \varepsilon E$. The other statement follows from b).

f) $(E \otimes F)^P = \text{Hom}(E \otimes F, K) = \text{Hom}(E, \text{Hom}(F, K)) \cong \text{Hom}((E^P)^P, F^P) = E^P \varepsilon F^P$.

g) By 18.9, there exists a natural transformation

from $\text{Hom}(E, F) \otimes \text{Hom}(G, H)$ to $\text{Hom}(E \otimes G, F \otimes H)$. So precomposing with $E \mapsto E^P$ and $G \mapsto G^P$, we get a natural transformation from $\text{Hom}(E^P, F) \otimes \text{Hom}(G^P, H)$ to $\text{Hom}(E^P \otimes G^P, F \otimes H)$. By e), $E^P \otimes G^P \cong (E \varepsilon G)^P$. So there exists a natural transformation from $\text{Hom}(E^P, F) \otimes \text{Hom}(G^P, H)$ to $\text{Hom}((E \varepsilon G)^P, F \otimes H)$, i.e. from $(E \varepsilon F) \otimes (G \varepsilon H)$ to $(E \varepsilon G) \varepsilon (F \otimes H)$.

h) This is just the dual statement of g), by taking duals and using e) and f).

i) $(E^P)^P \otimes F \mapsto \text{Hom}(E^P, F) = E \varepsilon F$ by 18.11. But $E^{PP} \cong E$. So $E \otimes F \mapsto E \varepsilon F$.

QED

Section 20 - \otimes and ε products of Frechet and dF spaces.

I prove that \otimes and ε take pairs of Frechet and pairs of dF spaces to Frechet and dF spaces respectively.

I prove that if E and F are either both Frechet or both dF, then precompact subsets of $E \otimes F$ are contained in the closed, convex, balanced hull $A \otimes B$ for some A compact in E and B compact in F .

Finally I show that if E and F are both Frechet spaces or both dF spaces, then my " ε "-product agrees with that of Schwartz and my " \otimes "-product agrees with Grothendieck's projective tensor product.

20.1 Theorem. If E and F are both Frechet spaces, then $E \varepsilon F$ and $E \otimes F$ are Frechet. If E and F are both dF spaces, then both $E \varepsilon F$ and $E \otimes F$ are dF spaces.

Proof: A) $(E \otimes F)^P = \text{Hom}(E \otimes F, \mathbb{K}) \cong \text{Hom}(E, \text{Hom}(F, \mathbb{K})) = \text{Hom}(E, F^P)$.

So if E and F are Frechet, $\text{Hom}(E, F^P)$ is a dF space by 8.8 and 8.10. Thus $E \otimes F \cong (E \otimes F)^{PP}$ is a Frechet space. If E and F are dF, then $\text{Hom}(E, F^P)$ is Frechet by 8.1 and 8.2. Hence $E \otimes F \cong (E \otimes F)^{PP}$ is a dF space.

B) $E \in F = \text{Hom}(E^{\mathbb{P}}, F)$. So if E and F are Frechet, then $E \in F$ is Frechet by 8.1 and 8.2. If E and F are dF spaces, then $E \in F$ is a dF space by 8.8 and 8.10. QED

20.2 Theorem. Suppose either E and F are both Frechet spaces, or E and F are both dF spaces. Then $A \subset E \otimes F$ is precompact iff there exists a compact set $B \subset E$ and a compact set $C \subset F$ such that A is contained in the closed, balanced, convex hull of $B \otimes C$ (cf. 17.12).

Proof: Recall from 17.8 that $E \otimes F = [B(E, F)]^{\mathbb{P}}$. Recall from 17.4 that $b_p(E, F) \cong \text{hom}_p(E, \text{hom}_p(F, \mathbb{K})) = \text{hom}_p(E, F^{\mathbb{P}})$. Thus the hypothesis, 8.1, 8.8, 6.2, and 6.3 guarantee that $b_p(E, F)$ is p -determined. So $B(E, F) = b_p(E, F)$ and hence $E \otimes F = [b_p(E, F)]^{\mathbb{P}}$.

(\Rightarrow) Let $A \subset E \otimes F$ be precompact. Then A is an equicontinuous subset of $[b_p(E, F)]'$. So A° is a neighborhood of zero in $b_p(E, F)$ and hence there exist compact sets $B \subset E$ and $C \subset F$ such that $A^\circ \supset \{b \in b(E, F) : |b(x, y)| \leq 1 \text{ for all } (x, y) \in B \times C\}$.

Let $a \in A$ and $b \in (B \otimes C)^\circ$. Now this implies that for all $(x, y) \in B \times C$, $1 \geq |(x \otimes y)(b)| = |b(x, y)|$. Thus by the above $b \in A^\circ$. So $|a(b)| \leq 1$. Thus $A \subset (B \otimes C)^{\circ\circ} =$ the closed, convex, balanced hull of $B \otimes C$.

(\Leftarrow) By 18.2, the canonical bilinear map from $E \times F$ to $E \otimes F$ is hypocontinuous. Hence if B and C are

precompact in E and F respectively, $B \otimes C$ is precompact in $E \otimes F$ (cf. Prop. 5, Chap. 3, §4, of [4]). Hence the closed, balanced, convex hull of $B \otimes C$ is precompact. QED

20.3 Theorem. Suppose that both E and F are Frechet spaces or both E and F are dF spaces. Then $E \otimes F \cong E \hat{\otimes}_{\pi} F$, where $E \hat{\otimes}_{\pi} F$ is the completed projective tensor product of Grothendieck.

Proof: Let $p: E \times F \rightarrow E \otimes F$ be the canonical bilinear form of 18.2. Let G be a complete locally convex space and $r: E \times F \rightarrow G$ be a continuous bilinear form. Then since this implies that r is hypocontinuous, we know that there exists a unique continuous linear map $\bar{r}: E \otimes F \rightarrow G$ such that $\bar{r} \circ p = r$.

By uniqueness up to isomorphism of the complete locally convex spaces and continuous bilinear forms which are solutions to the above universal problem; if I can show that $E \otimes F$ is complete and $p: E \times F \rightarrow E \otimes F$ is continuous, then I will have shown that $E \otimes F \cong E \hat{\otimes}_{\pi} F$.

But $E \otimes F$ is either a Frechet space or a dF space and is hence complete by 6.4. Also Prop. 2 of Chap. 3, §4 of [4] and 9.1 tell us that p is continuous. QED

20.4 Remark. In the event that E and F are both Banach spaces, then the topology of $E \hat{\otimes}_{\pi} F$ is the same as that induced by the "greatest cross-norm" of Schatten [37].

20.5 Theorem. Suppose E and F are both Frechet spaces or E and F are both dF spaces. Then $E \varepsilon F$ agrees with the " ε "-product of E and F of Schwartz (cf. [40]), and $E \hat{\otimes}_{\varepsilon} F$ is isomorphic to the closure of the algebraic tensor product of E and F in $\text{Hom}(E^{\mathcal{P}}, F) = E \varepsilon F$, where $E \hat{\otimes}_{\varepsilon} F$ denotes the tensor product of Grothendieck introduced on page 89 of [17]. Furthermore if either E or F satisfies the approximation property, then $E \hat{\otimes}_{\varepsilon} F \cong E \varepsilon F$.

Proof: If E and F are as in the hypothesis, then by 8.1 and 8.8, $\text{hom}_p(E^{\mathcal{P}}, F)$ is p -determined and hence $\text{hom}_p(E^{\mathcal{P}}, F) = \text{Hom}(E^{\mathcal{P}}, F) = E \varepsilon F$.

Let E'_C denote the continuous dual of E with the topology of uniform convergence on compact, convex sets. Next observe that equicontinuous^{sets} of E' are precompact in $E^{\mathcal{P}}$, and since E is p -determined, the converse is true as well. Hence $\text{hom}_p(E^{\mathcal{P}}, F) = \mathcal{L}_{\varepsilon}(E'_C, F)$, the latter thing being what Schwartz calls the space of continuous linear functions from E'_C to F with the topology of uniform convergence on equicontinuous subsets of E' .

Now Frechet and dF spaces are complete, therefore the result up to the "Furthermore" follows from proposition 5 of exposé 8 of [42] and from page 34 of [40].

The "Furthermore" part follows from the definition of page 8 and theorem 2 both of expose no. 14 of [42].

QED

20.6 Remark. In the event E and F are both Banach

spaces, then the topology of $E \hat{\otimes}_\varepsilon F$ is that of the "least cross-norm" of Schatten [37].

20.7 Corollary to theorem 20.1. If E and F are both Banach spaces, then $E \varepsilon F$ and $E \otimes F$ are Banach spaces. If E and F are both dB spaces, then $E \varepsilon F$ and $E \otimes F$ are dB spaces.

Proof: The proof is exactly the same as the proof of 20.1 except that "Banach" should be substituted for "Frechet", "dB" should be substituted for "dF", and references to 8.8 should be replaced by references to 8.9.

QED

Section 21 - The complexification of a real p -reflexive topological vector space.

The main theorem of this section is 21.9. Among other things it says that if E is a real p -reflexive space, then there exists a complex p -reflexive space $E_{\mathbb{C}}$ and a real linear continuous map $\rho_E: E \rightarrow E_{\mathbb{C}}$ such that if F is any complex p -reflexive space and $T: E \rightarrow F$ is a continuous real linear map, then there exists a unique complex linear map $\bar{T}: E_{\mathbb{C}} \rightarrow F$ such that $\bar{T} \circ \rho_E = T$.

Also I show that $(E \otimes_{\mathbb{R}} F)_{\mathbb{C}} \cong E_{\mathbb{C}} \otimes_{\mathbb{C}} F_{\mathbb{C}}$. And if $M_{\mathbb{R}}$ and $M_{\mathbb{C}}$ denote the functors of 10.5 for the different scalars, then $[M_{\mathbb{R}}(X)]_{\mathbb{C}} \cong M_{\mathbb{C}}(X)$ for all k -spaces X .

21.1 Definition. Suppose E is a locally convex Hausdorff space over \mathbb{R} . Further suppose that E is p -reflexive. Define a continuous biadditive function J from $\mathbb{C} \times (\mathbb{C} \otimes_{\mathbb{R}} E)$ to $\mathbb{C} \otimes_{\mathbb{R}} E$ as follows:

Let $j: \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \rightarrow \mathbb{C}$ be the continuous linear map induced by multiplication in the complex numbers. We get $j \otimes_{\mathbb{R}} 1_E: (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{R}} E \rightarrow \mathbb{C} \otimes_{\mathbb{R}} E$. But $(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{R}} E \cong \mathbb{C} \otimes_{\mathbb{R}} (\mathbb{C} \otimes_{\mathbb{R}} E)$. So we get a continuous real linear map from $\mathbb{C} \otimes_{\mathbb{R}} (\mathbb{C} \otimes_{\mathbb{R}} E)$ to $\mathbb{C} \otimes_{\mathbb{R}} E$. Let $J: \mathbb{C} \times (\mathbb{C} \otimes_{\mathbb{R}} E) \rightarrow \mathbb{C} \otimes_{\mathbb{R}} E$ be the hypocontinuous real bilinear map induced by this map.

In order to show that J is actually continuous,

let U be a neighborhood of zero in $\mathbb{C} \otimes_{\mathbb{R}} E$. Since D the closed unit disc in \mathbb{C} is compact, there exists a neighborhood V of zero in $\mathbb{C} \otimes_{\mathbb{R}} E$ such that $J(D \times V) \subset U$. Hence J is continuous at zero. But then by proposition 4.7.1 of [19], J is continuous. QED

21.2 Theorem. The abelian topological group structure from $\mathbb{C} \otimes_{\mathbb{R}} E$ together with the biadditive map J makes $\mathbb{C} \otimes_{\mathbb{R}} E$ into a p -reflexive topological vector space over \mathbb{C} .

Proof:

1) $J|(\mathbb{R} \times (\mathbb{C} \otimes_{\mathbb{R}} E))$ is just the scalar multiplication map for the real vector space $\mathbb{C} \otimes_{\mathbb{R}} E$.

Proof: Let $r \in \mathbb{R}$. It suffices to show that the continuous real linear maps $x \mapsto J(r, x)$ and $x \mapsto rx$ agree. To do that it will suffice to check the maps on elementary tensors. But $J(r, s \otimes x) = (rs) \otimes x = r(s \otimes x)$ for all $s \in \mathbb{C}$ and $x \in E$, by the definition of J .

2) If a and $b \in \mathbb{C}$ and $x \in \mathbb{C} \otimes_{\mathbb{R}} E$, then $J(a, J(b, x)) = J(ab, x)$.

Proof: Let a and $b \in \mathbb{C}$. It will suffice to show that the continuous linear maps $x \mapsto J(a, J(b, x))$ and $x \mapsto J(ab, x)$ agree on elementary tensors. So suppose $c \in \mathbb{C}$ and $e \in E$. But $J(a, J(b, c \otimes e)) = J(a, bc \otimes e) = a(bc) \otimes e = (ab)c \otimes e = J(ab, c \otimes e)$.

3) If the addition and topology are taken from the real vector space $\mathbb{C} \otimes_{\mathbb{R}} E$ and the scalar multiplication is taken from J , then we get a locally convex Hausdorff

topological vector space over \mathbb{C} .

Proof: J is biadditive and 2) above shows that it is a complex vector space, since $J(1,x) = x$ for all $x \in \mathbb{C} \otimes_{\mathbb{R}} E$ follows from 1) above. Also 1) together with the fact that $\mathbb{C} \otimes_{\mathbb{R}} E$ is a real locally convex Hausdorff space proves the rest.

4) The locally convex space described in 3) is p -reflexive.

Proof: 1) above makes this trivial since the real space $\mathbb{C} \otimes_{\mathbb{R}} E$ is p -reflexive and hence p -determined and p -complete. QED

21.3 Definition. If E is a real p -reflexive space, then call the complex vector space constructed on $\mathbb{C} \otimes_{\mathbb{R}} E$ in 21.2 the complexification of E and denote it by $E_{\mathbb{C}}$. Define a continuous real linear map $\rho_E: E \rightarrow E_{\mathbb{C}}$ by $e \mapsto 1 \otimes e$. Call ρ_E the canonical map from $E \rightarrow E_{\mathbb{C}}$.

21.4 Theorem. Let F be any complex p -complete space. Suppose $T: E \rightarrow F$ is continuous and real linear. Then there exists a unique continuous complex linear map $\bar{T}: E_{\mathbb{C}} \rightarrow F$ such that

$$\begin{array}{ccc}
 E & \xrightarrow{T} & F \\
 \rho \downarrow & \nearrow \bar{T} & \\
 E_{\mathbb{C}} & &
 \end{array}$$

commutes.

Proof: Let $1_{\mathbb{C}}: \mathbb{C} \rightarrow \mathbb{C}$ be the identity map regarded as a real linear map. Let $q: \mathbb{C} \times F \rightarrow F$ be the scalar

multiplication map for F . In particular q is continuous. So $q \circ (l_{\mathbb{C}} \times T): \mathbb{C} \times E \rightarrow F$ is a continuous real bilinear map into F . So by 18.2, this induces a real continuous map $\bar{T}: \mathbb{C} \otimes_{\mathbb{R}} E \rightarrow F$ such that $\bar{T}(c \otimes_{\mathbb{R}} e) = q((l_{\mathbb{C}} \times T)(c, e)) = cT(e)$ for all $(c, e) \in \mathbb{C} \times E$. [Note that F is p -complete regarded as a real linear space.]

1) \bar{T} is also complex linear.

Proof: Let $c \in \mathbb{C}$. It will be sufficient to show that the continuous real linear maps $x \mapsto \bar{T}(J(c, x))$ and $x \mapsto c[\bar{T}(x)]$ agree. In order to do that it suffices to check on elementary tensors. So let $d \in \mathbb{C}$ and $e \in E$. Then

$$\bar{T}(J(c, d \otimes e)) = \bar{T}(cd \otimes e) = cd[T(e)] = c(d[T(e)]) = c[\bar{T}(d \otimes e)].$$

2) $\bar{T} \circ \rho_E = T$.

Proof: $[\bar{T} \circ \rho_E](e) = \bar{T}(l \otimes e) = l[T(e)] = T(e)$ for all $e \in E$.

3) Suppose $S: E_{\mathbb{C}} \rightarrow F$ is a continuous complex linear map such that $S \circ \rho_E = T$, then $S = \bar{T}$.

Proof: Suppose $c \in \mathbb{C}$ and $e \in E$. Then $S(c \otimes e) = S(c(l \otimes e)) = c[S(l \otimes e)] = c[S \circ \rho_E(e)] = c[T(e)] = \bar{T}(c \otimes e)$. So S and \bar{T} agree on elementary tensors. Hence $S = \bar{T}$. QED

21.5 Lemma. If E is a complex locally convex Hausdorff space and if E is p -determined (resp., p -complete, or p -reflexive), then E regarded as a real locally convex Hausdorff space is p -determined (resp., p -complete, or p -reflexive).

Proof: p -completeness is trivial.

Now closedness and convexity (a real property) are the only things that count in p -determinedness by 1.7. Hence p -determinedness is trivial. p -reflexivity follows from the other two. QED

21.6 Convention. If a category or a functor has been defined relative to the field \mathbb{K} , then by putting a subscript \mathbb{R} or \mathbb{C} after the category or functor, I will mean that category or functor with \mathbb{R} or \mathbb{C} respectively substituted for \mathbb{K} everywhere.

21.7 Lemma. There exists a functor $F: \mathcal{D}_{\mathbb{C}} \rightarrow \mathcal{D}_{\mathbb{R}}$ such that if $E \in \text{ob } \mathcal{D}_{\mathbb{C}}$ then $F(E) = E$ regarded as a real vector space and if $T \in \text{Mor } \mathcal{D}_{\mathbb{C}}$ then $F(T) = T$ regarded as a real linear map.

Proof: 21.5.

21.8 Lemma. There exists a functor $G: \mathcal{D}_{\mathbb{R}} \rightarrow \mathcal{D}_{\mathbb{C}}$ such that if $E \in \text{ob } \mathcal{D}_{\mathbb{R}}$, then $G(E) = E_{\mathbb{C}}$ as defined in 21.3; and if $T \in \text{hom}_{\mathbb{R}}(E_1, E_2)$, then $G(T) =$ the unique continuous complex linear map which makes

$$\begin{array}{ccc}
 E_1 & \xrightarrow{T} & E_2 \\
 \rho \downarrow & & \downarrow \rho \\
 (E_1)_{\mathbb{C}} & \dashrightarrow & (E_2)_{\mathbb{C}}
 \end{array} \quad \text{commute.}$$

Proof: 21.4 and some trivial abstract nonsense. QED

21.9 Theorem. The functors $(A, B) \mapsto F(\text{Hom}_{\mathbb{C}}(G(A), B))$ and $(A, B) \mapsto \text{Hom}_{\mathbb{R}}(A, F(B))$ from $\mathcal{D}_{\mathbb{R}}^{\text{op}} \times \mathcal{D}_{\mathbb{C}}$ to $\mathcal{D}_{\mathbb{R}}$ are

naturally isomorphic.

21.10 Corollary. G is an adjoint of F . QED

Proof of 21.9: Suppose $Q \in \mathcal{D}_{\mathbb{R}}$ and $S \in \mathcal{D}_{\mathbb{C}}$. Define $\delta_{QS}: F[\text{Hom}_{\mathbb{C}}(G(Q), S)] \rightarrow \text{Hom}_{\mathbb{R}}(Q, F(S))$ by $\delta_{QS}(T) = T \circ \rho$ and $\eta_{QS}: \text{Hom}_{\mathbb{R}}(Q, F(S)) \rightarrow F[\text{Hom}_{\mathbb{C}}(G(Q), S)]$ by $\eta_{QS}(T) = \bar{T}$ where \bar{T} is the unique complex continuous linear map making

$$\begin{array}{ccc}
 Q & \xrightarrow{T} & S \\
 \rho \downarrow & \nearrow & \\
 Q_{\mathbb{C}} & &
 \end{array}$$

commute.

By 6.4.3 of [18] and 21.4, δ and η are natural isomorphisms, if the functors $\text{Hom}_{\mathbb{R}}$ and $\text{Hom}_{\mathbb{C}}$ are regarded as taking their values in the category of sets. So because the various forgetful functors into the category of sets are faithful, the theorem will be proved if I can demonstrate that for all $(Q, S) \in \mathcal{D}_{\mathbb{R}}^{\text{OP}} \times \mathcal{D}_{\mathbb{C}}$, δ_{QS} and η_{QS} are morphisms in the category $\mathcal{D}_{\mathbb{R}}$.

In the remainder of this proof I will omit most of the subscripts.

1) δ is real linear.

Proof: Suppose $T, T' \in \text{Hom}_{\mathbb{C}}(G(Q), S)$ and $\alpha, \beta \in \mathbb{R}$. Then $\delta(\alpha T + \beta T') = (\alpha T + \beta T') \circ \rho = \alpha(T \circ \rho) + \beta(T' \circ \rho) = \alpha\delta(T) + \beta\delta(T')$.

2) η is real linear.

Proof: $\delta^{-1} = \eta$ as functions and δ is real linear.

21.9a Technical lemma. Let $f: \mathcal{B}_{\mathbb{C}} \rightarrow \mathcal{B}_{\mathbb{R}}$ and

$F: \mathcal{D}_{\mathbb{C}} \rightarrow \mathcal{D}_{\mathbb{R}}$ be the functors that forget the complex

scalar multiplication. Then $F \circ \alpha_{\mathbb{C}} = \alpha_{\mathbb{R}} \circ f$ as functors from $\mathcal{D}_{\mathbb{C}}$ to $\mathcal{D}_{\mathbb{R}}$.

Proof of 21.9a: Suppose $E \in \mathcal{D}_{\mathbb{C}}$. Let V be the underlying vector space of E . Now $F(\alpha_{\mathbb{C}}(E))$ has a p -determined locally convex topology on V finer than $f(E)$ (cf. 21.5). So the topology of $F(\alpha_{\mathbb{C}}(E))$ is finer than that of $\alpha_{\mathbb{R}}(f(E))$ by 2.1.

On the other hand, let $T: \alpha_{\mathbb{R}}(f(E)) \rightarrow E$ be the identity map, considered as a real linear map. It is continuous because the topologies of $f(E)$ and E are the same. Let $\bar{T}: [\alpha_{\mathbb{R}}(f(E))]_{\mathbb{C}} \rightarrow E$ be the unique continuous complex linear map making

$$\begin{array}{ccc}
 \alpha_{\mathbb{R}}(f(E)) & \xrightarrow{T} & E \\
 \downarrow \rho & \nearrow \bar{T} & \\
 [\alpha_{\mathbb{R}}(f(E))]_{\mathbb{C}} & &
 \end{array}
 \quad \text{commute.}$$

Since $[\alpha_{\mathbb{R}}(f(E))]_{\mathbb{C}} \in \text{ob} \mathcal{D}_{\mathbb{C}}$, by 2.1 \bar{T} is actually continuous into $\alpha_{\mathbb{C}}(E)$. And since $\bar{T} \circ \rho = T$, T is actually continuous into $\alpha_{\mathbb{C}}(E)$. But the topologies of $F(\alpha_{\mathbb{C}}(E))$ and $\alpha_{\mathbb{C}}(E)$ are the same. So the topology of $\alpha_{\mathbb{R}}(f(E))$ is finer than that of $F(\alpha_{\mathbb{C}}(E))$. Thus $\alpha_{\mathbb{R}}(f(E)) = F(\alpha_{\mathbb{C}}(E))$. Thus on objects $\alpha_{\mathbb{R}} \circ f = F \circ \alpha_{\mathbb{C}}$. But none of the above functors alter morphisms, hence the two composites agree on morphisms as well. QED

Continuing with the proof of 21.9.

3) δ is continuous.

Proof: Suppose $T_\alpha \longrightarrow 0$ in $f([\text{hom}_p]_{\mathbb{C}}(G(Q), S))$. Then $T_\alpha \circ \rho \longrightarrow 0$ uniformly on precompact sets of Q . So δ is continuous from $f([\text{hom}_p]_{\mathbb{C}}(G(Q), S))$ to $[\text{hom}_p]_{\mathbb{R}}(Q, F(S))$. So δ is continuous from $\alpha_{\mathbb{R}}(f([\text{hom}_p]_{\mathbb{C}}(G(Q), S)))$ to $\alpha_{\mathbb{R}}([\text{hom}_p]_{\mathbb{R}}(Q, F(S)))$. Thus using 21.9a and the definitions of $\text{Hom}_{\mathbb{C}}$ and $\text{Hom}_{\mathbb{R}}$, δ is continuous from $F(\text{Hom}_{\mathbb{C}}(G(Q), S))$ to $\text{Hom}_{\mathbb{R}}(Q, F(S))$

4) η is continuous.

Proof: Define a map ϕ from $[\text{hom}_p]_{\mathbb{R}}(Q, F(S))$ to $[\text{hom}_p]_{\mathbb{R}}(\mathbb{C}, [\text{hom}_p]_{\mathbb{R}}(Q, F(S)))$ by $\phi(T)(c)(q) = c[T(q)]$.

Claim: ϕ is continuous.

Proof of claim: Suppose V is a balanced neighborhood of zero in $F(S)$ and M and L are precompact in \mathbb{C} and Q respectively. Let $\lambda > 0$ such that $k \in M$ implies $|k| < \lambda$. Then $\phi(N(Q, F(S); L, \lambda^{-1}V))$ is contained in $N(\mathbb{C}, [\text{hom}_p]_{\mathbb{R}}(Q, F(S)); M, N(Q, F(S); L, V))$.

So ϕ is continuous.

QED on claim

Using the last claim, if we apply $\alpha_{\mathbb{R}}$ to ϕ , by 17.5 and the definition of Hom we get that ϕ is continuous from $\text{Hom}_{\mathbb{R}}(Q, F(S))$ to $\text{Hom}_{\mathbb{R}}(\mathbb{C}, \text{Hom}_{\mathbb{R}}(Q, F(S)))$.

But $\text{Hom}_{\mathbb{R}}(\mathbb{C}, \text{Hom}_{\mathbb{R}}(Q, F(S))) \cong \text{Hom}_{\mathbb{R}}(\mathbb{C} \otimes_{\mathbb{R}} Q, F(S)) = \text{Hom}_{\mathbb{R}}(F(Q_{\mathbb{C}}), F(S))$.

Claim: if $T \in \text{hom}_{\mathbb{R}}(Q, F(S))$, then $\phi(T)$ considered as being in $\text{Hom}_{\mathbb{R}}(F(Q_{\mathbb{C}}), F(S))$ is complex linear from $Q_{\mathbb{C}}$ to S .

Proof: If we fix $c \in \mathbb{C}$, then $\phi(T)(J(c, d\otimes e)) = \phi(T)(cd\otimes e) = cd[T(e)] = c(d[T(e)]) = c[\phi(T)(d\otimes e)] = [c\phi(T)](d\otimes e)$ for all $d \in \mathbb{C}$ and $e \in S$. Thus on ele-

mentary tensors the two continuous real linear maps defined by $x \mapsto c[\phi(T)(x)]$ and $x \mapsto \phi(T)(cx)$ agree. Hence they must agree everywhere. But this same trick can be done for each $c \in \mathbb{C}$. Thus $\phi(T)$ is complex linear. Hence $\phi(T) \in \text{hom}_{\mathbb{C}}(Q_{\mathbb{C}}, S)$. QED on claim

Note that $\phi(T) = \eta(T)$. So η is continuous from $\text{Hom}_{\mathbb{R}}(Q, F(S))$ to $[\text{hom}_{\mathbb{P}}]_{\mathbb{R}}(F(Q_{\mathbb{C}}), F(S))$, and hence into $f([\text{hom}_{\mathbb{P}}]_{\mathbb{C}}(Q_{\mathbb{C}}, S))$ by the last claim. Thus η is continuous into $\alpha_{\mathbb{R}}(f([\text{hom}_{\mathbb{P}}]_{\mathbb{C}}(Q_{\mathbb{C}}, S)))$ because $\text{Hom}_{\mathbb{R}}(Q, F(S))$ is p -determined. But $\alpha_{\mathbb{R}} \circ f = F \circ \alpha_{\mathbb{C}}$ by 21.9a. So η is continuous into $F(\alpha_{\mathbb{C}}([\text{hom}_{\mathbb{P}}]_{\mathbb{C}}(Q_{\mathbb{C}}, S)))$ which equals $F(\text{Hom}_{\mathbb{C}}(G(Q), S))$. QED on 4)

QED

21.11 Corollary. Let $G: \mathcal{A}_{\mathbb{R}} \rightarrow \mathcal{A}_{\mathbb{C}}$ be the complexification functor as in 21.8. Then the functors $(E, F) \mapsto G(E \otimes_{\mathbb{R}} F)$ and $(E, F) \mapsto G(E) \otimes_{\mathbb{C}} G(F)$ from $\mathcal{A}_{\mathbb{R}} \times \mathcal{A}_{\mathbb{R}}$ to $\mathcal{A}_{\mathbb{C}}$ are naturally isomorphic.

Proof: Let L and M be in $\text{ob } \mathcal{A}_{\mathbb{R}}$ and N be in $\text{ob } \mathcal{A}_{\mathbb{C}}$.

$$\begin{aligned} \text{Then } F(\text{Hom}_{\mathbb{C}}(G(L) \otimes_{\mathbb{C}} G(M), N)) &\cong F(\text{Hom}_{\mathbb{C}}(G(L), \text{Hom}_{\mathbb{C}}(G(M), N))) \\ &\cong \text{Hom}_{\mathbb{R}}(L, F(\text{Hom}_{\mathbb{C}}(G(M), N))) \\ &\cong \text{Hom}_{\mathbb{R}}(L, \text{Hom}_{\mathbb{R}}(M, F(N))) \\ &\cong \text{Hom}_{\mathbb{R}}(L \otimes_{\mathbb{R}} M, F(N)) \\ &\cong F(\text{Hom}_{\mathbb{C}}(G(L \otimes_{\mathbb{R}} M), N)) \end{aligned}$$

by repeated applications of 17.11 and 21.9. Thus

$\text{hom}_{\mathbb{C}}(G(L) \otimes_{\mathbb{C}} G(M), N)$ and $\text{hom}_{\mathbb{C}}(G(L \otimes_{\mathbb{R}} M), N)$ are naturally isomorphic. So by A5, $G(L) \otimes_{\mathbb{C}} G(M)$ is naturally isomorphic to $G(L \otimes_{\mathbb{R}} M)$. QED

21.12 Proposition. Let $M_{\mathbb{R}}: \mathcal{J} \rightarrow \mathcal{D}_{\mathbb{R}}$ and $M_{\mathbb{C}}: \mathcal{J} \rightarrow \mathcal{D}_{\mathbb{C}}$ denote the functors described in 10.5 for real and complex scalars respectively. Let $G: \mathcal{D}_{\mathbb{R}} \rightarrow \mathcal{D}_{\mathbb{C}}$ be the complexification functor described in 21.8. Then $G \circ M_{\mathbb{R}}$ is naturally isomorphic to $M_{\mathbb{C}}$.

Proof: Let $X \in \text{ob } \mathcal{J}$ and $E \in \text{ob } \mathcal{D}_{\mathbb{C}}$. Then

$$\begin{aligned} \text{hom}_{\mathbb{C}}(G(M_{\mathbb{R}}(X)), E) &\cong \text{hom}_{\mathbb{R}}(M_{\mathbb{R}}(X), F(E)) \\ &\cong \text{Mor}_{\mathcal{J}}(X, \gamma_{\mathbb{R}}(F(E))) \quad \text{by 10.6} \\ &\cong \text{Mor}_{\mathcal{J}}(X, \gamma_{\mathbb{C}}(E)) \quad \text{because 21.7} \\ &\cong \text{hom}_{\mathbb{C}}(M_{\mathbb{C}}(X), E) \quad \text{by 10.6.} \end{aligned}$$

Hence $\text{hom}_{\mathbb{C}}(G \circ M_{\mathbb{R}}(X), E)$ is naturally isomorphic to $\text{hom}_{\mathbb{C}}(M_{\mathbb{C}}(X), E)$. Hence by A5, $G \circ M_{\mathbb{R}}$ is naturally isomorphic to $M_{\mathbb{C}}$. QED

Section 22 - The conjugate space functor from \mathcal{C} to \mathcal{C} .

What follows is a short discussion of the
 functor from \mathcal{C} to \mathcal{C} induced by the complex
 conjugation automorphism from \mathbb{C} to \mathbb{C} .

22.1 Definition. Define a functor $H: \mathcal{C} \rightarrow \mathcal{C}$ as follows: if $E \in \text{ob } \mathcal{C}$, then $H(E)$ is the underlying set of E with the same additive structure and topology as E , but with scalar multiplication defined by $c \cdot e = \bar{c}e$ where the bar denotes complex conjugation and juxtaposition denotes the scalar multiplication of E ; and if E and F are in $\text{ob } \mathcal{C}$ and $f \in \text{Mor}_{\mathcal{C}}(E, F)$, then $H(f) = f$.
 H will be called the conjugate space functor.

22.2 Remark. If $\mathbb{K} = \mathbb{R}$, then H is merely the identity functor on \mathcal{C} . Hence all of what follows is trivial in this case.

22.3 Proposition. H is well-defined.

Proof: Since as topological groups E and $H(E)$ agree, the only thing that needs to be ^{shown} in order that $H(E)$ be a topological vector space is the continuity of the scalar multiplication. But it is continuous since the map $c \mapsto \bar{c}$ from \mathbb{C} to \mathbb{C} is continuous.

Now since complex conjugation leaves real numbers fixed, scalar multiplication by real numbers is the same

in $H(E)$ as in E . Hence the convex sets of E and of $H(E)$ agree. Thus if E is locally convex, so is $H(E)$.

The rest of the verification that H is a functor is quite easy. QED

22.4 Lemma. If E is a p -determined (resp., p -complete or p -reflexive) space, then so is $H(E)$.

Proof: As noted in the proof of 22.3, the convex subsets of E and of $H(E)$ agree as do the topologies. So if E is p -determined, by using 1.7, we see that $H(E)$ is also. Since as abelian topological groups $H(E) = E$, the two uniformities also agree. Hence if E is p -complete, then so is $H(E)$. The fact that if E is p -reflexive, then $H(E)$ is p -reflexive, follows from the other two facts. QED

22.5 Remark. Due to 22.4, H can be regarded as a functor from \mathcal{O}_1 to \mathcal{O}_1 , from \mathcal{E} to \mathcal{E} , or from \mathcal{D} to \mathcal{D} .

22.6 Proposition. Let I_e denote the identity functor on \mathcal{C} . Then $H \circ H = I_e$.

Proof: trivial.

22.7 Theorem. The functors $(E, F) \mapsto H(\text{Hom}(E, F))$ and $(E, F) \mapsto \text{Hom}(H(E), H(F))$ from $\mathcal{D}^{\text{op}} \times \mathcal{D}$ to \mathcal{D} are equal.

Proof: Let us begin by checking the equality for objects. Let $E \in \text{ob } \mathcal{D}$ and $F \in \text{ob } \mathcal{D}$.

It is easily verified that as sets,

$\text{hom}_p(E, F) = \text{hom}_p(H(E), H(F))$; and hence as sets

$$H(\text{hom}_p(E, F)) = \text{hom}_p(H(E), H(F)).$$

It is easy to check that the additive group structure of $\text{hom}_p(E, F)$ and $\text{hom}_p(H(E), H(F))$ agree, and hence the additive structure of $H(\text{hom}_p(E, F))$ agrees with that of $\text{hom}_p(H(E), H(F))$.

Suppose $T \in H(\text{hom}_p(E, F))$ and $\alpha \in K$. Then $\alpha \cdot T = \bar{\alpha}T$. So if $e \in E$, then $(\alpha \cdot T)(e) = (\bar{\alpha}T)(e) = \bar{\alpha}(T(e))$. But also if $T \in \text{hom}_p(H(E), H(F))$, then $(\alpha T)(e) = \alpha \cdot (T(e)) = \bar{\alpha}(T(e))$ for all $e \in E$. Hence the scalar multiplication on $H(\text{hom}_p(E, F))$ and $\text{hom}_p(H(E), H(F))$ agree.

Hence as vector spaces $H(\text{hom}_p(E, F))$ equals $\text{hom}_p(H(E), H(F))$.

The uniformities of $H(E)$ and $H(F)$ agree with the uniformities of E and F respectively, and hence the topologies of $\text{hom}_p(E, F)$ and $\text{hom}_p(H(E), H(F))$ agree. Thus the topologies of $H(\text{hom}_p(E, F))$ and $\text{hom}_p(H(E), H(F))$ agree. So as topological vector spaces these two spaces agree.

Now as vector spaces, $H(\alpha(\text{hom}_p(E, F))) = H(\text{hom}_p(E, F)) = \text{hom}_p(H(E), H(F))$. But the topology of $H(\alpha(\text{hom}_p(E, F)))$ is a p -determined topology finer than the topology of the topological vector space $H(\text{hom}_p(E, F)) = \text{hom}_p(H(E), H(F))$. Hence the topology of $H(\text{Hom}(E, F))$ is finer than that of $\text{Hom}(H(E), H(F))$ (cf. the definition of Hom and 2.1).

Substituting $H(E)$ for E and $H(F)$ for F , and recalling that $H \circ H = I_e$, we get that

$H(\text{Hom}(H(E), H(F)))$ is finer than $\text{Hom}(E, F)$. Applying H again, we see that $\text{Hom}(H(E), H(E))$ is finer than $H(\text{Hom}(E, F))$.

So $\text{Hom}(H(E), H(F)) = H(\text{Hom}(E, F))$ as topological vector spaces.

That $\text{Hom} \circ (H^{\text{op}} \times H)$ and $H \circ \text{Hom}$ agree on morphisms is easily verified. QED

22.8 Theorem. The functors $(E, F) \mapsto H(E) \otimes H(F)$ and $(E, F) \mapsto H(E \otimes F)$ from $\mathcal{S} \times \mathcal{S}$ to \mathcal{S} are naturally isomorphic.

Proof: Let $E, F,$ and G be in $\text{ob } \mathcal{S}$. Then

$$\begin{aligned} \text{Hom}(H(E \otimes F), G) &= \text{Hom}(H(E \otimes F), H(H(G))) \quad \text{by 22.6} \\ &= H(\text{Hom}(E \otimes F, H(G))) \quad \text{by 22.7} \\ &\cong H(\text{Hom}(E, \text{Hom}(F, H(G)))) \quad \text{by 17.11} \\ &= \text{Hom}(H(E), H(\text{Hom}(F, H(G)))) \quad \text{by 22.7} \\ &= \text{Hom}(H(E), \text{Hom}(H(F), H(H(G)))) \quad \text{by 22.7} \\ &= \text{Hom}(H(E), \text{Hom}(H(F), G)) \quad \text{by 22.6} \\ &\cong \text{Hom}(H(E) \otimes H(F), G) \quad \text{by 17.11} \end{aligned}$$

So $\text{Hom}(H(E \otimes F), G)$ is naturally isomorphic to $\text{Hom}(H(E) \otimes H(F), G)$. Thus by A5, $H(\cdot) \otimes H(\cdot)$ is naturally isomorphic to $H(\cdot \otimes \cdot)$. QED

Chapter Four

M , \square , \otimes , Multiplication, and Comultiplication

The notions of k -monoid and k -group are introduced. These are generalizations of topological semi-group with identity and topological group respectively.

The natural definitions of \mathcal{D} -algebra, \mathcal{D} -coalgebra, \mathcal{D} -bialgebra, and \mathcal{D} -Hopf algebra are given, using the \mathcal{D} -tensor product in place of the algebraic one.

It is shown that for all $X, Y \in \text{ob } \mathcal{J}$, $M(X) \otimes M(Y)$ is naturally isomorphic to $M(X \sqcup Y)$ and that $M(\{1\})$ is isomorphic to \mathbb{K} . These two facts together with the fact that M is a functor are what make everything in this chapter go. In particular, using this information one can prove that M can be regarded as a functor from k -spaces to \mathcal{D} -coalgebras, from k -monoids to \mathcal{D} -bialgebras, and from k -groups to \mathcal{D} -Hopf algebras.

As in the purely algebraic case, one defines a functor Ω , the group-like element functor, from \mathcal{D} -coalgebras to k -spaces, from \mathcal{D} -bialgebras to k -monoids, and from \mathcal{D} -Hopf algebras to k -groups. In each of these three situations, it is shown that Ω is a coadjoint to M . In addition, if X is a topologically p -complete k -space, then X is isomorphic to $\Omega(M(X))$.

With regard to representation theory, it is shown that if G is a k -monoid, then the category of representations of the k -monoid G is isomorphic to the category of representations of the \mathcal{D} -algebra $M(G)$. Also if G is a k -group, then $M(G)$ can be shown to have the structure of a \mathcal{D} -algebra-with-involution. In this case, the category of unitary Hilbert space representations of the

k -group G is shown to be isomorphic to the category of "star" Hilbert space representations of the \mathcal{O} -algebra-with-involution $M(G)$.

Thus it is possible that $M(G)$ will aid in the study of the representation theory of topological groups which are not locally compact, but which are k -spaces; for example, metrizable groups. Also the comultiplication on $M(G)$ may be helpful in the study of "inner" tensor products of group representations.

Finally, let me mention for the reader who is interested in category theory that almost everything in this chapter is a consequence of the fact that both \mathcal{J} and \mathcal{O} are symmetric, monoidal categories, and that $M: \mathcal{J} \rightarrow \mathcal{O}$ is a "strong", symmetric, monoidal functor (cf. [14]).

Section 23 - The interaction between \mathcal{J} and \mathcal{D} .

The interaction between \mathcal{J} and \mathcal{D} is perhaps best described in terms of the notion of monoidal functor. But in order to discuss this, it is first necessary to have the concept of monoidal category.

Monoidal categories turn out to be the proper categorical setting to define many algebraic concepts, such as algebras, Hopf algebras, and groups.

Following is a brief description of some of these basic notions. For a more extensive treatment of these matters see [9], [13], [14], [24], [28], and [30].

23.1 Definition. By a symmetric monoidal category we shall mean the following collection of data:

- i) a category \mathcal{C} ;
- ii) a functor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$;
- iii) an object K of \mathcal{C} ;
- iv) natural isomorphisms

$$a = a_{ABC}: A \otimes (B \otimes C) \longrightarrow (A \otimes B) \otimes C,$$

$$l = l_A: K \otimes A \longrightarrow A,$$

$$r = r_A: A \otimes K \longrightarrow A,$$

$$c = c_{AB}: A \otimes B \longrightarrow B \otimes A.$$

The axioms to be satisfied by these data are that, for all $A, B, C, D \in \text{ob}\mathcal{C}$, the following four diagrams should commute:

C1

$$\begin{array}{ccccc}
 A \otimes (B \otimes (C \otimes D)) & \xrightarrow{a} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{a} & ((A \otimes B) \otimes C) \otimes D \\
 \downarrow l \otimes a & & & & \uparrow a \otimes l \\
 A \otimes ((B \otimes C) \otimes D) & \xrightarrow{a} & (A \otimes (B \otimes C)) \otimes D & &
 \end{array}$$

C2

$$\begin{array}{ccc}
 A \otimes (K \otimes B) & \xrightarrow{a} & (A \otimes K) \otimes B \\
 \downarrow l \otimes l & \searrow & \swarrow r \otimes l \\
 & A \otimes B &
 \end{array}$$

C3

$$\begin{array}{ccc}
 A \otimes B & \xrightarrow{l} & A \otimes B \\
 \downarrow c & & \uparrow c \\
 & B \otimes A &
 \end{array}$$

C4

$$\begin{array}{ccccc}
 A \otimes (B \otimes C) & \xrightarrow{a} & (A \otimes B) \otimes C & \xrightarrow{c} & C \otimes (A \otimes B) \\
 \downarrow l \otimes c & & & & \downarrow a \\
 A \otimes (C \otimes B) & \xrightarrow{a} & (A \otimes C) \otimes B & \xrightarrow{c \otimes l} & (C \otimes A) \otimes B
 \end{array}$$

23.2 Remark. Such categories are discussed in [14] and [30]. Axioms C1 through C4 insure that a , l , r , and c are coherent in the sense of [23], [25], [30], and [31]. Roughly speaking, this means that any diagram will commute if (as in the diagrams of axioms C1 through C4) each arrow is a natural isomorphism manufactured from l_K , $id_C: I_C \rightarrow I_C$ (where I_C is the identity functor on \mathcal{C}), a , a^{-1} , r , r^{-1} , l , l^{-1} , and c by taking repeated \otimes -products*. Note that coherence asserts equality of natural transformations, and not of morphisms in \mathcal{C} except insofar as these are components of natural trans-

* - Note that $id_C(A) = l_A$ for all $A \in \text{ob } \mathcal{C}$.

formations; thus it does not assert that $c_{AA}: A \otimes A \rightarrow A \otimes A$ and $\text{id}_c(A) \otimes \text{id}_c(A): A \otimes A \rightarrow A \otimes A$ coincide, these being components of quite different natural transformations $c: A \otimes B \rightarrow B \otimes A$ and $\text{id}_c \otimes \text{id}_c: A \otimes B \rightarrow A \otimes B$.

23.3 Definition. If $\underline{\mathcal{C}} = (\mathcal{C}, \otimes, K, a, \ell, r, c)$ and $\underline{\mathcal{C}}' = (\mathcal{C}', \otimes', K', a', \ell', r', c')$ are symmetric monoidal categories, then by a strong symmetric monoidal functor from $\underline{\mathcal{C}}$ to $\underline{\mathcal{C}}'$ I shall mean the following collection of data:

- i) a functor $\phi: \mathcal{C} \rightarrow \mathcal{C}'$;
- ii) a natural isomorphism $q = q_{AB}: \phi(A) \otimes' \phi(B) \rightarrow \phi(A \otimes B)$;
- iii) an isomorphism $j: K' \rightarrow \phi(K)$.

The axioms to be satisfied by these data are that, for all $A, B, C \in \text{ob } \mathcal{C}$, the following four diagrams should commute:

F1

$$\begin{array}{ccc}
 \phi(A) \otimes' (\phi(B) \otimes' \phi(C)) & \xrightarrow{a'} & (\phi(A) \otimes' \phi(B)) \otimes' \phi(C) \\
 \downarrow 1 \otimes q & & \downarrow q \otimes 1 \\
 \phi(A) \otimes' \phi(B \otimes C) & & \phi(A \otimes B) \otimes' \phi(C) \\
 \downarrow q & & \downarrow q \\
 \phi(A \otimes (B \otimes C)) & \xrightarrow{\phi(a)} & \phi((A \otimes B) \otimes C)
 \end{array}$$

F2

$$\begin{array}{ccc}
 K' \otimes' \phi(A) & \xrightarrow{\ell'} & \phi(A) \\
 \downarrow j \otimes 1 & & \uparrow \phi(\ell) \\
 \phi(K) \otimes' \phi(A) & \xrightarrow{q} & \phi(K \otimes A)
 \end{array}$$

F3

$$\begin{array}{ccc}
 \phi(A) \otimes' K' & \xrightarrow{r'} & \phi(A) \\
 \downarrow l \otimes j & & \uparrow \phi(r) \\
 \phi(A) \otimes' \phi(K) & \xrightarrow{q} & \phi(A \otimes K)
 \end{array}$$

F4

$$\begin{array}{ccc}
 \phi(A) \otimes' \phi(B) & \xrightarrow{c'} & \phi(B) \otimes' \phi(A) \\
 \downarrow q & & \downarrow q \\
 \phi(A \otimes B) & \xrightarrow{\phi(c)} & \phi(B \otimes A)
 \end{array}$$

23.4 Remark. If one didn't insist on having isomorphisms in ii) and iii) of 23.3, then that which would result is called a symmetric monoidal functor in [14].

23.5 Definition. For a symmetric monoidal category $\underline{\mathcal{C}}$ we can construct by suitable combinations of a , a^{-1} , and c (the details being irrelevant by coherence), a natural isomorphism

$$m = m_{ABCD}: (A \otimes B) \otimes (C \otimes D) \longrightarrow (A \otimes C) \otimes (B \otimes D),$$

called the middle-four exchange (cf. 23.2).

We shall have need for this concept later in our discussion of bialgebras.

23.6 Proposition. Let \mathcal{J} be the category of k -spaces and continuous maps. Let $\square: \mathcal{J} \times \mathcal{J} \rightarrow \mathcal{J}$ be the k -space product functor (cf. 16.0). Then there exist natural isomorphisms

$$l' = l'_X: \{1\} \square X \longrightarrow X;$$

$$r' = r'_X: X \square \{1\} \longrightarrow X;$$

$$c' = c'_{XY}: X \sqcap Y \longrightarrow Y \sqcap X;$$

$$a' = a'_{XYZ}: X \sqcap (Y \sqcap Z) \longrightarrow (X \sqcap Y) \sqcap Z.$$

Moreover these natural isomorphisms are defined by

$$(1, x) \longmapsto x,$$

$$(x, 1) \longmapsto x,$$

$$(x, y) \longmapsto (y, x), \text{ and}$$

$$(x, (y, z)) \longmapsto ((x, y), z) \text{ respectively.}$$

Note: See [45] for a discussion of the product on \mathcal{J} .

Proof: Let \mathcal{J}_0 be the category of Hausdorff topological spaces and continuous maps. Let $k: \mathcal{J}_0 \longrightarrow \mathcal{J}$ denote the coreflection functor. Let $\times: \mathcal{J}_0 \times \mathcal{J}_0 \longrightarrow \mathcal{J}_0$ denote the usual product functor. Then if X and Y are k -spaces, then $X \sqcap Y = k(X \times Y)$. Also if $X, Y \in \text{ob}\mathcal{J}_0$, then $k(X \times Y) = k(k(X) \times k(Y))$ (cf. [45], lemma 4.5).

Using these facts and familiar facts about the functor \times , then above results are easy to verify. QED

In fact

23.7 Theorem. $(\mathcal{J}, \sqcap, \{1\}, a', \ell', r', c')$ is a symmetric monoidal category. (Also it is a closed symmetric monoidal category in the sense of [13] or [30]).

Proof: Using 23.6, it is easy to verify that the axioms of 23.1 are satisfied. Alternatively see page 183 of [30] and/or [45]. QED

23.8 Theorem. Using the notation of 18.3,

$(\mathcal{D}, \otimes, \mathbb{K}, a, \ell, r, c)$ is a symmetric monoidal category. (Also it is a closed symmetric monoidal category in the sense of [13] or [30]).

Proof: In 18.3, we demonstrated that the four natural isomorphisms of the definition of symmetric monoidal category exist. Let us now check that axioms C1 through C4 are satisfied.

In each case, due to 18.1 it is sufficient to check the commutativity of the diagrams on elementary tensors. But on elementary tensors the fact that each of the diagrams commute is trivial by 18.3, i.e. by the manner in which each of the natural isomorphisms work.

Also for all $B \in \text{ob } \mathcal{D}$, the functor $- \otimes B$ has a coadjoint (cf. 18.5). Hence by the definition of [13] and [30], \mathcal{D} is a closed symmetric monoidal category.

QED

We have shown that both \mathcal{J} and \mathcal{D} are symmetric monoidal categories. In fact $M: \mathcal{J} \rightarrow \mathcal{D}$ is a strong symmetric monoidal functor.

The following group of propositions each lead up to this fact.

23.9 Theorem. The functors $(X, Y) \mapsto M(X) \otimes M(Y)$ and $(X, Y) \mapsto M(X \boxplus Y)$ from $\mathcal{J} \times \mathcal{J}$ to \mathcal{D} are naturally isomorphic.

Proof: Composing M a couple of times with the functors in 17.11 and by using A1, we find that

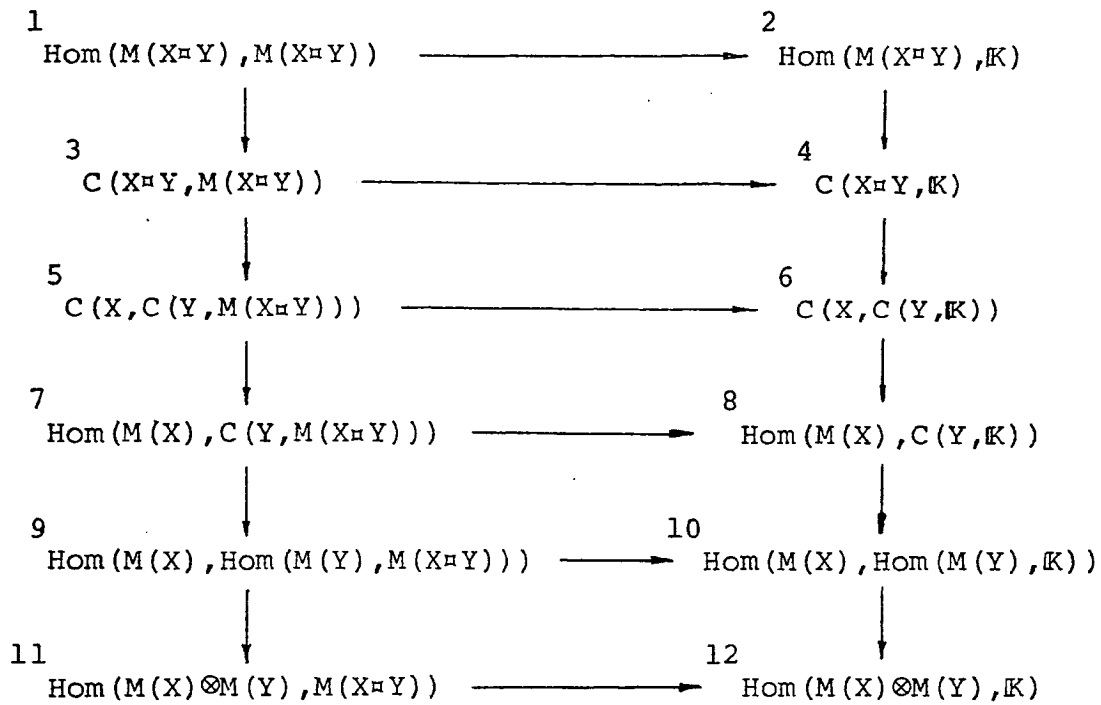
$\text{Hom}(M(-) \otimes M(-), -)$ is naturally isomorphic to $\text{Hom}(M(-), \text{Hom}(M(-), -))$ as functors from $\mathcal{J}^{\text{op}} \times \mathcal{J}^{\text{op}} \times \mathcal{D}$ to \mathcal{D} . By 16.7, $\text{Hom}(M(-), \text{Hom}(M(-), -))$ is naturally isomorphic to $\text{Hom}(M(- \boxplus -), -)$.

So composing we get that $\text{Hom}(M(-) \otimes M(-), -)$ is naturally isomorphic to $\text{Hom}(M(- \boxplus -), -)$. Applying the forgetful functor to both sides and using A5, we conclude that $M(- \boxplus -)$ and $M(-) \otimes M(-)$ are naturally isomorphic. QED

For computations it will be useful to know that

23.10 Theorem. Let $q = q_{XY}: M(X) \otimes M(Y) \longrightarrow M(X \boxplus Y)$ be the natural isomorphism of 23.9. Let X and Y be k -spaces. Then if $f \in c(X \boxplus Y)$, $\mu \in M(X)$, and $\nu \in M(Y)$, we have $[q(\mu \otimes \nu)](f) = \nu(y \longmapsto \mu(x \longmapsto f(x, y)))$.

Proof: Let f , μ , and ν be as above. Now f can be regarded as a continuous linear function from $M(X \boxplus Y)$ to \mathbb{K} , i.e. $\mu \longmapsto \mu(f)$. So because $\text{Hom}(M(-), -)$ is naturally isomorphic to $C(-, -)$ by 10.6, we get a series of commuting diagrams as follows:



Now $1_{M(X \boxplus Y)}$ in the upper left hand corner (object #1) is transformed by the various arrows to elements of each of the various objects labeled 1 through 12. For each i , $1 \leq i \leq 12$, let $R(i)$ denote the image of $1_{M(X \boxplus Y)}$ under these various arrows at stage i . Thus, for example, $R(1) = 1_{M(X \boxplus Y)}$; and by 23.9, $R(11) = q$. Also $[q(\mu \otimes v)](f) = [R(12)](\mu \otimes v)$. Note that $R(4) = f$.

Define $f_1: X \rightarrow C(Y)$ and $f_2: Y \rightarrow C(X)$ by $f_1(x)(y) = f(x, y) = f_2(y)(x)$. Then $R(8) = \overline{f_1}$, where $\overline{f_1}$ is the unique continuous linear map from $M(X)$ to $C(Y)$ such that $\overline{f_1} \circ \varepsilon = f_1$.

At stage 10, $R(10) = \mu \mapsto (v \mapsto [\overline{f_1}(\mu)](v))$, where $[\overline{f_1}(\mu)]$ is the unique continuous linear map from $M(Y)$ to \mathbb{K} such that $[\overline{f_1}(\mu)] \circ \varepsilon = \overline{f_1}(\mu)$. But by 10.7, $[\overline{f_1}(\mu)](v) = v(\overline{f_1}(\mu))$; and by 16.8, $\overline{f_1}(\mu) = \mu \circ f_2$.

Thus at stage 12, we get that

$$[q(\mu \otimes \nu)](f) = R(12)(\mu \otimes \nu) = \nu(\mu \circ f_2). \quad \text{QED}$$

23.11 Corollary. Let X and Y be k -spaces. Let $q: M(X) \otimes M(Y) \rightarrow M(X \sqcup Y)$ be the natural isomorphism. Then the map $r: X \sqcup Y \rightarrow M(X) \otimes M(Y)$ defined by $r(x, y) = \varepsilon(x) \otimes \varepsilon(y)$ is continuous and

$$\begin{array}{ccc}
 & X \sqcup Y & \\
 r \swarrow & & \searrow \varepsilon \\
 M(X) \otimes M(Y) & \xrightarrow{q} & M(X \sqcup Y)
 \end{array}
 \quad \text{commutes.}$$

Proof: The canonical bilinear form p from $M(X) \times M(Y)$ to $M(X) \otimes M(Y)$ is continuous when restricted to the products of compact sets (since it is hypocontinuous). Recall that if K is a compact subset of $X \times Y$, then on K the relative topologies induced by $X \times Y$ and by $X \sqcup Y$ agree. Thus we see that the map defined by $(x, y) \mapsto p(\varepsilon(x), \varepsilon(y))$ is continuous when restricted to the compact subsets of $X \sqcup Y$. Hence this map is continuous from $X \sqcup Y$ to $M(X) \otimes M(Y)$, since $X \sqcup Y$ is a k -space. Thus r is continuous. Let $(x, y) \in X \sqcup Y$. Let $f: X \sqcup Y \rightarrow \mathbb{K}$ be a continuous function. Then by 23.10,

$$[q(\varepsilon(x) \otimes \varepsilon(y))](f) = f(x, y) = [\varepsilon(x, y)](f). \quad \text{So}$$

$$(q \circ r)(x, y) = \varepsilon(x, y), \quad \text{and hence } q \circ r = \varepsilon. \quad \text{QED}$$

23.12 Proposition. If $\{1\}$ is a k -space with cardinality one and if $\varepsilon: \{1\} \rightarrow M(\{1\})$ is the canonical map, then the map j defined by $\alpha \mapsto \alpha[\varepsilon(1)]$ is a

\mathcal{C} -isomorphism from \mathbb{K} to $M(\{1\})$.

Proof: Let $f: \{1\} \rightarrow \mathbb{K}$ be defined by $f(1) = 1$. f is continuous, so there exists a continuous linear map $\bar{f}: M(\{1\}) \rightarrow \mathbb{K}$ such that $\bar{f} \circ \varepsilon = f$. If j is as in the hypothesis, then j is continuous and linear since $M(\{1\})$ is a topological vector space over \mathbb{K} .

Note that $(j \circ \bar{f}) \circ \varepsilon = j \circ f = \varepsilon$ and that $(\bar{f} \circ j) \circ f = \bar{f} \circ \varepsilon = f$. This implies that $j \circ \bar{f} = 1_{M(\{1\})}$ and $\bar{f} \circ j = 1_{\mathbb{K}}$. Hence j is a \mathcal{C} -isomorphism. QED

23.13 Proposition. Let q be the natural isomorphism described in 23.10. Then for all $X, Y, Z \in \text{obj } \mathcal{J}$, the following diagram commutes:

$$\begin{array}{ccccc}
 M(X) \otimes (M(Y) \otimes M(Z)) & \xrightarrow{1 \otimes q} & M(X) \otimes M(Y \sqcup Z) & \xrightarrow{q} & M(X \sqcup (Y \sqcup Z)) \\
 \downarrow a & & & & \downarrow M(a') \\
 (M(X) \otimes M(Y)) \otimes M(Z) & \xrightarrow{q \otimes 1} & M(X \sqcup Y) \otimes M(Z) & \xrightarrow{q} & M((X \sqcup Y) \sqcup Z)
 \end{array}$$

Proof: Since the linear span of the point evaluations of Y and Z are dense in $M(Y)$ and $M(Z)$ respectively by 10.6, we have that the linear span of $\{\varepsilon(y) \otimes \varepsilon(z) : y \in Y \text{ \& } z \in Z\}$ is dense in $M(Y) \otimes M(Z)$ by using 18.1. By the same reasoning the linear span of $\{\varepsilon(x) \otimes (\varepsilon(y) \otimes \varepsilon(z)) : x \in X, y \in Y, \text{ \& } z \in Z\}$ is dense in $M(X) \otimes (M(Y) \otimes M(Z))$. Therefore since everything in sight is continuous, it suffices to check the commutativity on elements of the form $\varepsilon(x) \otimes (\varepsilon(y) \otimes \varepsilon(z))$.

Thus by using 23.11 together with 18.3 and 23.6, it is trivial that the diagram commutes. QED

23.14 Proposition. Let q be the natural isomorphism described in 23.10, and j be the isomorphism described in 23.12. Then for all $X \in \text{ob}\mathcal{J}$, the following diagrams commute:

$$\begin{array}{ccc}
 M(X) \otimes \mathbb{K} & \xrightarrow{r} & M(X) \\
 \downarrow l \otimes j & & \uparrow M(r') \\
 M(X) \otimes M(\{1\}) & \xrightarrow{q} & M(X \sqcup \{1\})
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{K} \otimes M(X) & \xrightarrow{l} & M(X) \\
 \downarrow j \otimes 1 & & \uparrow M(l') \\
 M(\{1\}) \otimes M(X) & \xrightarrow{q} & M(\{1\} \sqcup X)
 \end{array}$$

Proof: The linear span of $\{\varepsilon(x) : x \in X\}$ is dense in $M(X)$ and the linear span of $\{1\}$ is dense in \mathbb{K} . So the linear span of $\{\varepsilon(x) \otimes 1 : x \in X\}$ and $\{1 \otimes \varepsilon(x) : x \in X\}$ is dense in $M(X) \otimes \mathbb{K}$ and $\mathbb{K} \otimes M(X)$ respectively. So it suffices to check the commutativity on these spanning sets.

Thus by 18.3, 23.6, and 23.11, the result follows. QED

23.15 Proposition. Let q be the natural isomorphism described in 23.10. Then for all $X, Y \in \text{ob}\mathcal{J}$, the following diagram commutes:

$$\begin{array}{ccc}
 M(X) \otimes M(Y) & \xrightarrow{q} & M(X \sqcup Y) \\
 \downarrow c & & \downarrow M(c') \\
 M(Y) \otimes M(X) & \xrightarrow{q} & M(Y \sqcup X)
 \end{array}$$

Proof: By 18.1, since the linear span of the point

evaluations of X and Y are dense in $M(X)$ and $M(Y)$ respectively by 10.6; it will suffice to check the commutativity on elements of the form $\varepsilon(x) \otimes \varepsilon(y)$ where $x \in X$ and $y \in Y$. Thus by 18.3, 23.6, and 23.11, the result follows. QED

23.16 Remark. The following is sort of a Fubini theorem.

23.17 Corollary. If q is the natural isomorphism described in 23.10, and if $X, Y \in \text{obj } \mathcal{J}$, $\mu \in M(X)$, $\nu \in M(Y)$, and $f \in C(X \sqcup Y)$, then $q: M(X) \otimes M(Y) \rightarrow M(X \sqcup Y)$ can be computed as follows:

$$\begin{aligned} [q(\mu \otimes \nu)](f) &= \nu(y \mapsto \mu(x \mapsto f(x, y))) \\ &= \mu(x \mapsto \nu(y \mapsto f(x, y))) \end{aligned}$$

Proof: Using the notation of 23.15, what we are interested in finding is $[q(\mu \otimes \nu)](f)$. But this is equal to $(M(c')^{-1}[q[c(\mu \otimes \nu)]])(f) = q(\nu \otimes \mu)(f \circ c')$ by 23.15. But this equals $\mu(x \mapsto \nu(y \mapsto f(c'(y, x))))$ by 23.10. Also $f(c'(y, x)) = f(x, y)$. So we have one half of the result. The other half follows from 23.10.

QED

23.18 Theorem. Let M be the functor described in 10.5, q be the natural isomorphism described in 23.10, and j be the isomorphism described in 23.12. Then (M, q, j) is a strong symmetric monoidal functor from $(\mathcal{J}, \sqcup, \{1\}, a', \ell', r', c')$ to $(\mathcal{D}, \otimes, \mathbb{K}, a, \ell, r, c)$.

Proof: 23.13, 23.14, and 23.15.

QED

23.19 Definition. For the symmetric monoidal categories $(\mathcal{J}, \sqsupset, \{1\}, a', \ell', r', c')$ and $(\mathcal{D}, \otimes, \mathbb{K}, a, \ell, r, c)$, let $m' = m'_{WXYZ}: (W \sqsupset X) \sqsupset (Y \sqsupset Z) \longrightarrow (W \sqsupset Y) \sqsupset (X \sqsupset Z)$ and $m = m_{ABCD}: (A \otimes B) \otimes (C \otimes D) \longrightarrow (A \otimes C) \otimes (B \otimes D)$ respectively be their middle-four exchanges (cf. 23.5).

23.20 Proposition. m' and m are defined by

$$\begin{aligned} ((w, x), (y, z)) &\longmapsto ((w, y), (x, z)) \quad \text{and} \\ (a \otimes b) \otimes (c \otimes d) &\longmapsto (a \otimes c) \otimes (b \otimes d) \quad \text{respectively.} \end{aligned}$$

Proof: The proof follows easily from 23.5, 23.6, and 18.3. QED

23.21 Theorem. For all $W, X, Y, Z \in \text{ob}\mathcal{J}$, the following diagram commutes:

$$\begin{array}{ccc} (M(W) \otimes M(X)) \otimes (M(Y) \otimes M(Z)) & \xrightarrow{m} & (M(W) \otimes M(Y)) \otimes (M(X) \otimes M(Z)) \\ \downarrow q \otimes q & & \downarrow q \otimes q \\ M(W \sqsupset X) \otimes M(Y \sqsupset Z) & & M(W \sqsupset Y) \otimes M(X \sqsupset Z) \\ \downarrow q & & \downarrow q \\ M((W \sqsupset X) \sqsupset (Y \sqsupset Z)) & \xrightarrow{M(m')} & M((W \sqsupset Y) \sqsupset (X \sqsupset Z)) \end{array}$$

Proof: Let ε be as in 10.5.

By 23.11, $q(\varepsilon(x) \otimes \varepsilon(y)) = \varepsilon(x, y)$ for all $x \in X$ and $y \in Y$. We also have that $M(m') \circ \varepsilon = \varepsilon \circ m'$. Now by 18.1, it is sufficient to check the diagram on tensors of the form $(\varepsilon(w) \otimes \varepsilon(x)) \otimes (\varepsilon(y) \otimes \varepsilon(z))$.

By using all of the above information it is easy to conclude that the diagram commutes. QED

23.22 Remark. Actually any symmetric monoidal functor has this property (i.e. 23.21). The abstract proof uses axioms F1 and F4.

Section 24 - \mathcal{D} -Hopf algebras.

In this section the categories of \mathcal{D} -coalgebras, \mathcal{D} -bialgebras, and \mathcal{D} -Hopf algebras are defined. It is also shown that in a natural way M can be regarded as a functor from k -spaces to \mathcal{D} -coalgebras, from " k -topological semigroups" to \mathcal{D} -bialgebras, and from " k -topological groups" to \mathcal{D} -Hopf algebras.

24.1 Definition. A triple (A, P, U) will be called a \mathcal{D} -algebra provided A is a p -reflexive locally convex topological vector space, P is a continuous linear map from $A \otimes A$ to A (called the multiplication map), U is a continuous linear map from \mathbb{K} to A (called the unit), and provided the following three diagrams commute:

$$\begin{array}{ccc}
 \text{a)} & & \text{b)} & & \text{c)} \\
 A \otimes (A \otimes A) & \xrightarrow{a} & (A \otimes A) \otimes A & & \mathbb{K} \otimes A & \xrightarrow{l} & A \\
 \downarrow l \otimes P & & \downarrow P \otimes 1 & & \downarrow U \otimes 1 & & \uparrow P \\
 A \otimes A & \xrightarrow{P} & A & & A \otimes A & & A \\
 & & & & & & \downarrow P \\
 & & & & & & A \otimes \mathbb{K} & \xrightarrow{r} & A \\
 & & & & & & \downarrow l \otimes U & & \uparrow P \\
 & & & & & & A \otimes A & &
 \end{array}$$

24.2 Definition. A \mathcal{D} -algebra (A, P, U) will be called commutative provided the following diagram commutes:

$$\begin{array}{ccc}
 A \otimes A & \xrightarrow{c} & A \otimes A \\
 \downarrow P & & \downarrow P \\
 & A &
 \end{array}$$

24.3 Definition. A triple (C, Δ, ζ) will be called a \mathcal{D} -coalgebra provided C is a p -reflexive locally convex topological vector space, Δ is a continuous linear map from C to $C \otimes C$ (called the comultiplication map), ζ is a continuous linear map from C to \mathbb{K} (called the counit), and provided the following three diagrams commute:

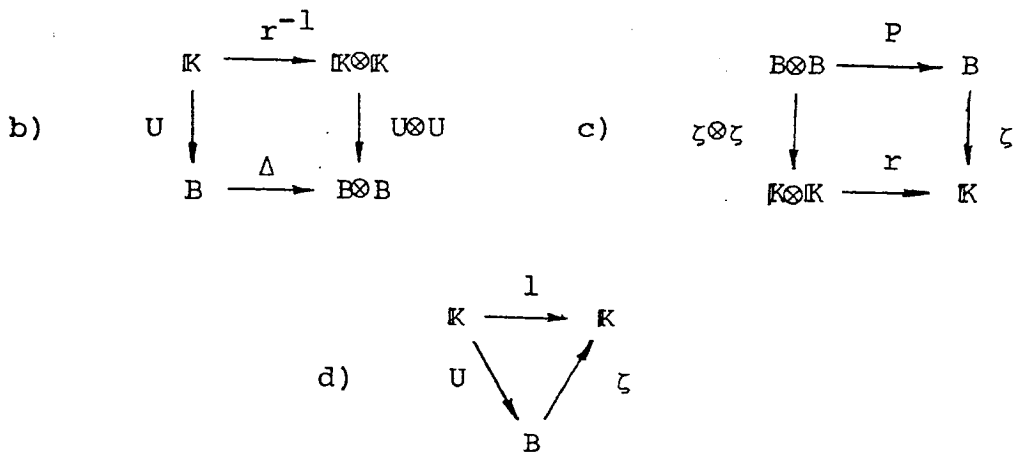
$$\begin{array}{ccc}
 \text{a)} & & \text{b)} \\
 \begin{array}{ccc}
 C \otimes C & \xleftarrow{\Delta} & C & \xrightarrow{\Delta} & C \otimes C \\
 \Delta \otimes 1 \downarrow & & & & \downarrow 1 \otimes \Delta \\
 (C \otimes C) \otimes C & \xrightarrow{a^{-1}} & C \otimes (C \otimes C) & &
 \end{array} & &
 \begin{array}{ccc}
 & C \otimes C & \\
 \Delta \nearrow & & \searrow \zeta \otimes 1 \\
 C & \xrightarrow{q^{-1}} & \mathbb{K} \otimes C
 \end{array} \\
 \end{array}
 \quad
 \begin{array}{ccc}
 \text{c)} & & \\
 \begin{array}{ccc}
 & C \otimes C & \\
 \Delta \nearrow & & \searrow 1 \otimes \zeta \\
 C & \xrightarrow{r^{-1}} & C \otimes \mathbb{K}
 \end{array}
 \end{array}$$

24.4 Definition. A \mathcal{D} -coalgebra (C, Δ, ζ) will be called cocommutative provided the following diagram commutes:

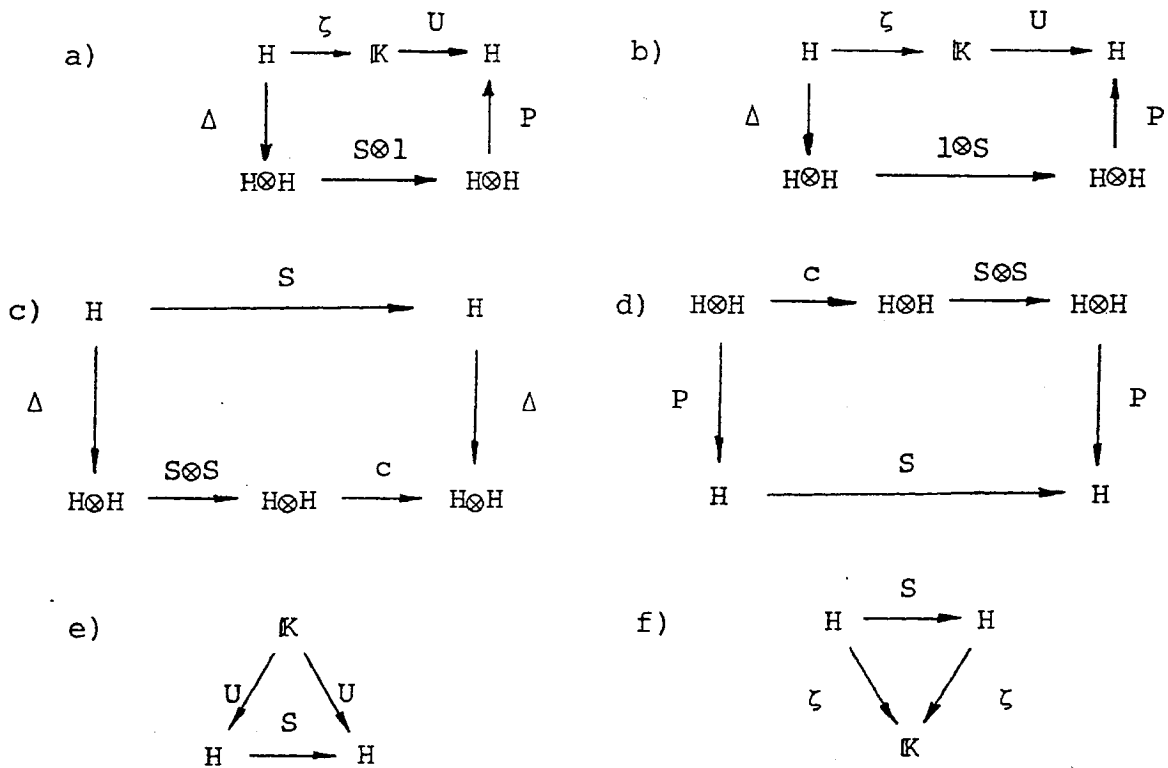
$$\begin{array}{ccc}
 & C & \\
 \Delta \swarrow & & \searrow \Delta \\
 C \otimes C & \xrightarrow{c} & C \otimes C
 \end{array}$$

24.5 Definition. A five-tuple (B, P, U, Δ, ζ) will be called a \mathcal{D} -bialgebra provided (B, P, U) is a \mathcal{D} -algebra, (B, Δ, ζ) is a \mathcal{D} -coalgebra, and provided that the following four diagrams commute:

$$\begin{array}{ccccc}
 & B \otimes B & \xrightarrow{P} & B & \xrightarrow{\Delta} & B \otimes B \\
 & \downarrow \Delta \otimes \Delta & & & & \uparrow P \otimes P \\
 \text{a)} & (B \otimes B) \otimes (B \otimes B) & \xrightarrow{m} & (B \otimes B) \otimes (B \otimes B) & &
 \end{array}$$



24.6 Definition. A six-tuple $(H, P, U, \Delta, \zeta, S)$ will be called a \mathcal{D} -Hopf algebra provided (H, P, U, Δ, ζ) is a \mathcal{D} -bialgebra, S is a continuous linear map from H to H (called the antipode), and provided that the following six diagrams commute:



24.7 Definition. A \mathcal{D} -bialgebra or a \mathcal{D} -Hopf algebra is called commutative provided its associated \mathcal{D} -algebra

is commutative.

24.8 Definition. A \mathcal{D} -bialgebra or a \mathcal{D} -Hopf algebra is called cocommutative provided its associated \mathcal{D} -coalgebra is cocommutative.

24.9 Definition. Let (A, P, U) and (A', P', U') be \mathcal{D} -algebras. Then a \mathcal{D} -algebra morphism from (A, P, U) to (A', P', U') will be any continuous linear map f from A to A' which makes the following two diagrams commute:

$$\text{a) } \begin{array}{ccc} A \otimes A & \xrightarrow{P} & A \\ f \otimes f \downarrow & & \downarrow f \\ A' \otimes A' & \xrightarrow{P'} & A' \end{array}$$

$$\text{b) } \begin{array}{ccc} & K & \\ U \swarrow & & \searrow U' \\ A & \xrightarrow{f} & A' \end{array}$$

24.10 Definition. Let (C, Δ, ζ) and (C', Δ', ζ') be \mathcal{D} -coalgebras. Then a \mathcal{D} -coalgebra morphism from (C, Δ, ζ) to (C', Δ', ζ') will be any continuous linear map f from C to C' which makes the following two diagrams commute:

$$\text{a) } \begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ f \downarrow & & \downarrow f \otimes f \\ C' & \xrightarrow{\Delta'} & C' \otimes C' \end{array}$$

$$\text{b) } \begin{array}{ccc} C & \xrightarrow{f} & C' \\ \zeta \searrow & & \swarrow \zeta' \\ & K & \end{array}$$

24.11 Definition. Let $\mathcal{B} = (B, P, U, \Delta, \zeta)$ and $\mathcal{B}' = (B', P', U', \Delta', \zeta')$ be \mathcal{D} -bialgebras. Then a \mathcal{D} -bialgebra morphism from \mathcal{B} to \mathcal{B}' will be any continuous linear map f from B to B' such that f is a \mathcal{D} -algebra morphism from (B, P, U) to (B', P', U') and

f is a \mathcal{D} -coalgebra morphism from (B, Δ, ζ) to (B', Δ', ζ') .

24.12. Definition. Let $\mathcal{H} = (H, P, U, \Delta, \zeta, S)$ and $\mathcal{H}' = (H', P', U', \Delta', \zeta', S')$ be \mathcal{D} -Hopf algebras. Then a continuous linear map f from H to H' will be called a \mathcal{D} -Hopf algebra morphism from \mathcal{H} to \mathcal{H}' provided f is a \mathcal{D} -bialgebra morphism from (H, P, U, Δ, ζ) to $(H', P', U', \Delta', \zeta')$ and provided that the following diagram commutes:

$$\begin{array}{ccc} H & \xrightarrow{f} & H' \\ S \downarrow & & \downarrow S' \\ H & \xrightarrow{f} & H' \end{array}$$

24.13 Facts. Since I do not intend to study \mathcal{D} -Hopf algebras in what follows, I shall list below some facts which while interesting are not needed in the sequel and hence will not be proved.

1) The tensor product of two \mathcal{D} -algebras has a natural \mathcal{D} -algebra structure and the tensor product of two \mathcal{D} -coalgebras has a natural \mathcal{D} -coalgebra structure.

2) The commutativity of diagrams a) and b) in the definition of \mathcal{D} -bialgebra states that Δ is a \mathcal{D} -algebra morphism. The commutativity of diagrams c) and d) in that definition states that ζ is a \mathcal{D} -algebra morphism. The commutativity of diagrams a) and c) states that P is a \mathcal{D} -coalgebra morphism. The commutativity of diagrams b) and d) states that U is a \mathcal{D} -coalgebra morphism.

3) The commutativity of diagrams c), d), e), and f) of the definition of \mathcal{D} -Hopf algebra states that S is both an anti-(algebra morphism) and an anti-(coalgebra morphism).

4) If (B, P, U, Δ, ζ) is a \mathcal{D} -bialgebra, then there exists at most one continuous linear map $S: B \rightarrow B$ such that diagrams a) and b) in the definition of \mathcal{D} -Hopf algebra commute. If $S: B \rightarrow B$ satisfies diagrams a) and b) in the definition of \mathcal{D} -Hopf algebra, then diagrams c), d), e), and f) in that definition are automatically satisfied.

5) If $(H, P, U, \Delta, \zeta, S)$ is a \mathcal{D} -Hopf algebra which is either commutative or cocommutative, then $S \circ S = 1_H$.

6) Let $\mathcal{H} = (H, P, U, \Delta, \zeta, S)$ and $\mathcal{H}' = (H', P', U', \Delta', \zeta', S')$ be \mathcal{D} -Hopf algebras. If f is a \mathcal{D} -bialgebra morphism from (H, P, U, Δ, ζ) to $(H', P', U', \Delta', \zeta')$, then f is a \mathcal{D} -Hopf algebra morphism from \mathcal{H} to \mathcal{H}' . End of 24.13

The following will provide us with an example of a \mathcal{D} -algebra.

24.14 Theorem.

1) For every $A, B, C \in \text{ob } \mathcal{D}$, there is a \mathcal{D} -morphism $O = O_{ABC}: \text{Hom}(B, C) \otimes \text{Hom}(A, B) \rightarrow \text{Hom}(A, C)$ such that $O(S \otimes T) = S \circ T$, where $S \circ T$ denotes the composition of S and T .

2) For every object $A \in \text{ob } \mathcal{D}$, there is a \mathcal{D} -morphism $U = U_A: K \rightarrow \text{Hom}(A, A)$ such that $U(\alpha) = \alpha 1_A$.

3) \circ and U are such that for every A, B, C, D in $\text{ob}\mathcal{D}$, the following three diagrams always commute:

$$\begin{array}{ccc}
 \text{Hom}(C, D) \otimes [\text{Hom}(B, C) \otimes \text{Hom}(A, B)] & \xrightarrow{a} & [\text{Hom}(C, D) \otimes \text{Hom}(B, C)] \otimes \text{Hom}(A, B) \\
 \downarrow 1 \otimes \circ & & \downarrow \circ \otimes 1 \\
 \text{Hom}(C, D) \otimes \text{Hom}(A, C) & \xrightarrow{\circ} & \text{Hom}(A, D) \xleftarrow{\circ} \text{Hom}(B, D) \otimes \text{Hom}(A, B)
 \end{array}$$

$$\begin{array}{ccc}
 \mathbb{K} \otimes \text{Hom}(A, B) & \xrightarrow{\ell} & \text{Hom}(A, B) \\
 \downarrow U \otimes 1 & & \uparrow \circ \\
 \text{Hom}(B, B) \otimes \text{Hom}(A, B) & &
 \end{array}$$

$$\begin{array}{ccc}
 \text{Hom}(A, B) \otimes \mathbb{K} & \xrightarrow{r} & \text{Hom}(A, B) \\
 \downarrow 1 \otimes U & & \uparrow \circ \\
 \text{Hom}(A, B) \otimes \text{Hom}(A, A) & &
 \end{array}$$

24.15 Corollary. For all $A \in \text{ob}\mathcal{D}$, $(\text{Hom}(A, A), \circ_{AAA}, U_A)$ is a \mathcal{D} -algebra. QED

Proof of 24.14: This theorem is a consequence of \mathcal{D} being a closed category as is proved in [9], [13], and [14]. I will give a direct proof.

Let $\pi = \pi_{ABC}: \text{hom}(A \otimes B, C) \rightarrow \text{hom}(A, \text{Hom}(B, C))$ be the canonical natural isomorphism. Let $t = t_{AB} = (\pi_{\text{Hom}(A, B)AB})^{-1}(1_{\text{Hom}(A, B)}): \text{Hom}(A, B) \otimes A \rightarrow B$.

Consider

$$\text{Hom}(B, C) \xrightarrow{\text{Hom}(t, 1)} \text{Hom}(\text{Hom}(A, B) \otimes A, C) \xrightarrow{\pi} \text{Hom}(\text{Hom}(A, B), \text{Hom}(A, C)).$$

Let $[\pi \circ \text{Hom}(t, l)]^*$ denote the element of $\text{hom}(\text{Hom}(B, C) \otimes \text{Hom}(A, B), \text{Hom}(A, C))$ which corresponds to $\pi \circ \text{Hom}(t, l)$. Let $S \in \text{Hom}(B, C)$, $T \in \text{Hom}(A, B)$, and $a \in A$. Then

$$\begin{aligned} [[\pi \circ \text{Hom}(t, l)]^*(S \otimes T)](a) &= \\ [[(\pi \circ \text{Hom}(t, l))(S)](T)](a) &= \\ [[\text{Hom}(t, l)](S)](T \otimes a) &= \\ (S \circ t)(T \otimes a) &= \\ S(t(T \otimes a)) &= \\ S(T(a)) &= (S \circ T)(a) \end{aligned}$$

So $[\pi \circ \text{Hom}(t, l)]^*(S \otimes T) = S \circ T$. Hence there is a continuous linear map from $\text{Hom}(B, C) \otimes \text{Hom}(A, B)$ to $\text{Hom}(A, C)$ such that $S \otimes T$ is sent to $S \circ T$. Thus define $O = (\pi \circ \text{Hom}(t, l))^*$.

Since it suffices to check these diagrams on elementary tensors, the fact that $O(S \otimes T) = S \circ T$ immediately yields the commutativity of the diagrams. QED

k -spaces, k -monoids, k -groups, and the action of M on them.

The notions of k -monoid and k -group, which will be introduced shortly, are generalizations of topological semigroup with identity and topological group respectively (that is they are generalizations in the inclusive sense, since I don't have any examples where these concepts are ^{strictly} more general). They are introduced not for their "generality", but rather because they fit naturally into the category theoretic framework in which I am

working. In particular, I am not sure if the functor Ω , to be introduced in section 25, can be defined without the notions of k -monoid and k -group.

24.16 Remark. Recall that if X and $Y \in \text{ob}\mathcal{J}$, then $X \sqsupset Y$ is the set $X \times Y$ with the coarsest k -space topology on $X \times Y$ which is finer than the product topology (cf. [45]). So if X and Y are both locally compact Hausdorff or if X and Y are both metrizable, then the topology of $X \sqsupset Y$ is simply the product topology.

24.17 Definition. Let G be a set that is a monoid (i.e. a semigroup with identity) and is also a topological space. Suppose that:

- i) G is a k -space; and
- ii) the semigroup multiplication map is continuous as a function from $G \sqsupset G$ to G .

Then G is called a k -monoid.

24.18 Definition. A group G which is a k -monoid is called a k -group if the map $x \mapsto x^{-1}$ is continuous.

24.19 Lemma. If G is a topological group (resp., topological semigroup with identity) which when considered as a topological space is a k -space, then G is a k -group (resp., k -monoid).

Proof: One only need note that the topology of $G \sqsupset G$ is finer than the product topology. QED

24.20 Definition. Let G and G' be k -monoids. Then

a function $f: G \rightarrow G'$ is a k-monoid morphism provided

- i) f is continuous; and
- ii) f is a semigroup homomorphism which preserves identities.

24.21 Definition. Let G and G' be k -groups. Then a function $f: G \rightarrow G'$ is a k-group morphism provided f is a k -monoid morphism.

24.22 Definition. We will call a \mathcal{D} -algebra, \mathcal{D} -bialgebra, or a \mathcal{D} -Hopf algebra non-zero provided its underlying topological vector space is non-zero.

24.23 Notation. Let \mathcal{L} = the category of non-zero \mathcal{D} -Hopf algebras and \mathcal{D} -Hopf algebra morphisms.

Let \mathcal{M} = the category of non-zero \mathcal{D} -bialgebras and \mathcal{D} -bialgebra morphisms.

Let \mathcal{N} = the category of \mathcal{D} -coalgebras and \mathcal{D} -coalgebra morphisms.

Let \mathcal{R} = the category of k -groups and k -group morphisms.

Let \mathcal{I} = the category of k -monoids and k -monoid morphisms.

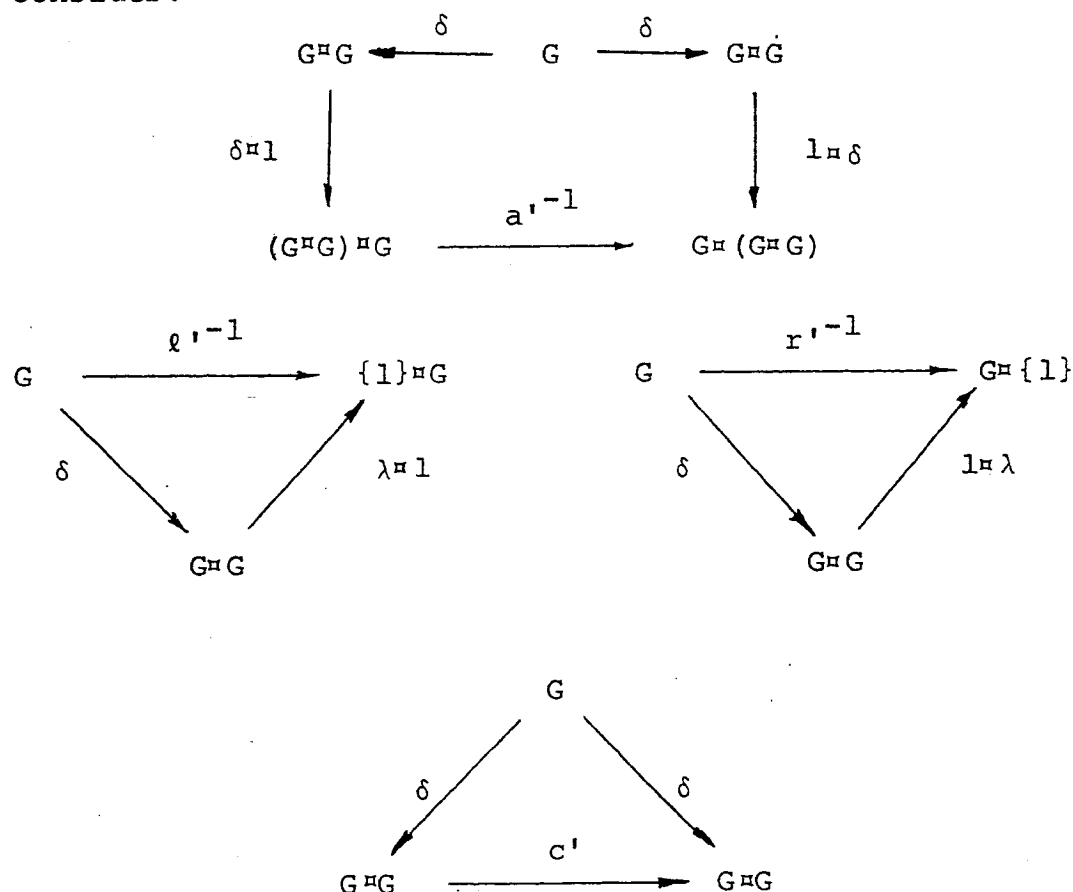
24.24 Theorem. If G is a k -group (k -monoid, or k -space respectively), then the structure of G induces on $M(G)$ the structure of a non-zero cocommutative \mathcal{D} -Hopf algebra (non-zero cocommutative \mathcal{D} -bialgebra, or cocommutative \mathcal{D} -coalgebra respectively) such that M is a functor from \mathcal{R} to \mathcal{L} (from \mathcal{I} to \mathcal{M} , or

from \mathcal{J} to \mathcal{K} respectively) if the identifications of 23.18 are made.

Proof: The proof will follow easily by noting certain facts about k -spaces, k -monoids, and k -groups; and by using the fact that (M, q, j) is a strong symmetric monoidal functor (cf. 23.18) between the symmetric monoidal categories $\underline{\mathcal{J}} = (\mathcal{J}, \square, \{1\}, a', \ell', r', c')$ and $\underline{\mathcal{D}} = (\mathcal{D}, \otimes, \mathbb{K}, a, \ell, r, c)$ [cf. 23.7 and 23.8].

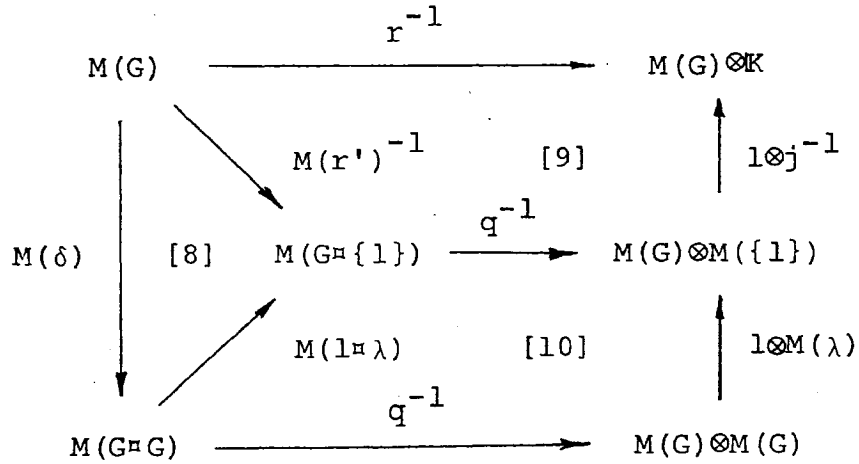
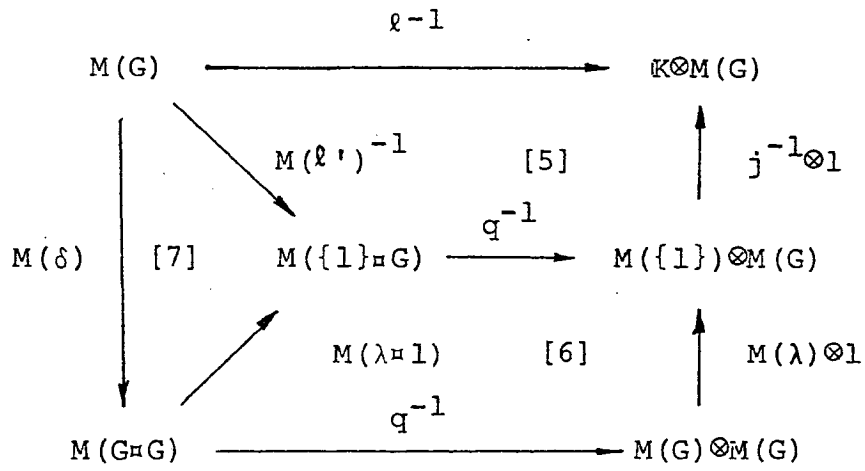
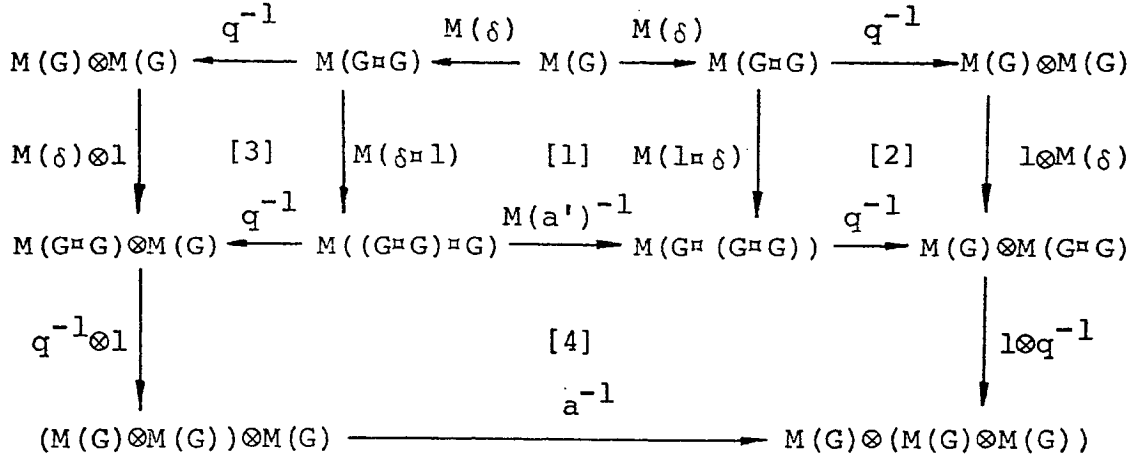
i) Let G be a k -space. Let $\delta: G \rightarrow G \square G$ and $\lambda: G \rightarrow \{1\}$ be defined by $\delta(g) = (g, g)$ and $\lambda(g) = 1$.

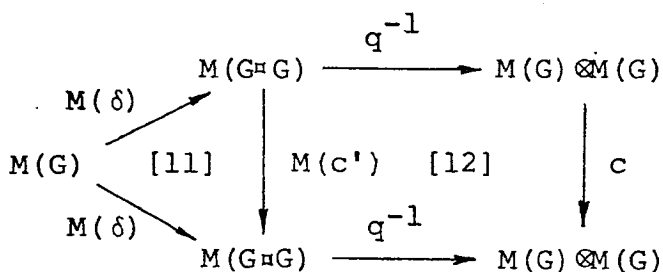
Consider:



Each of these four diagrams commute. Next consider the following four large diagrams, each consisting of various

"subdiagrams".

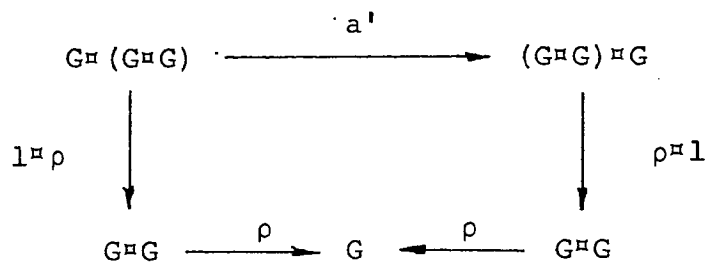


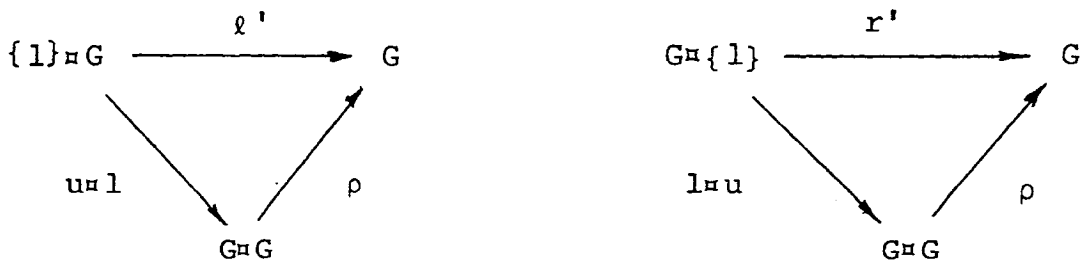


Diagrams [1], [7], [8], and [11] commute since M is a functor and since the first group of four diagrams commute. Diagrams [2], [3], [6], and [10] commute because q is natural. Diagrams [4], [5], [9], and [12] commute because (M, q, j) is a strong symmetric monoidal functor. Thus the four large diagrams (formed by the outside arrows) commute.

This proves that $(M(G), q^{-1} \circ M(\delta), j^{-1} \circ M(\lambda))$ is a cocommutative \mathcal{D} -coalgebra provided G is a k -space.

ii) Suppose now that G is a k -monoid. Let e be the identity of G . Let $\lambda: G \rightarrow \{1\}$ and $\delta: G \rightarrow G \boxplus G$ be as in i). Define $\rho: G \boxplus G \rightarrow G$ and $u: \{1\} \rightarrow G$ by $\rho(x, y) = xy$ and $u(1) = e$. Then the following three diagrams commute:



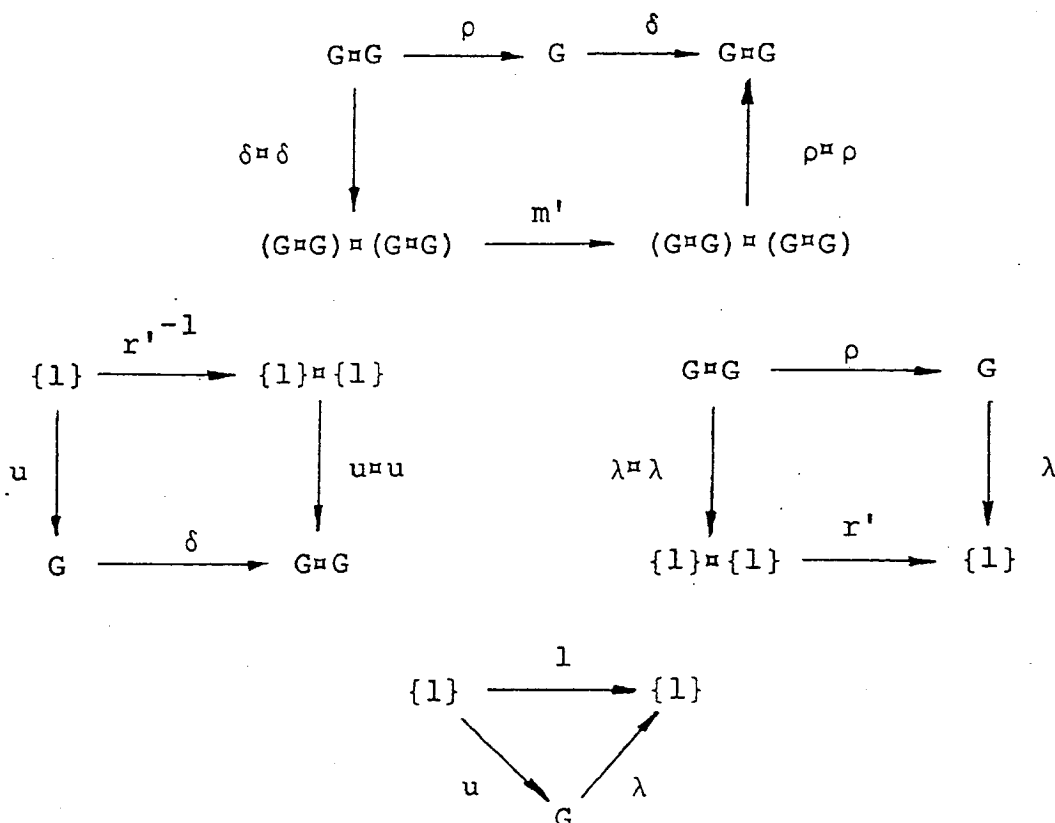


Using essentially the same proof as in i) except turning the arrows around; it is easy to see that

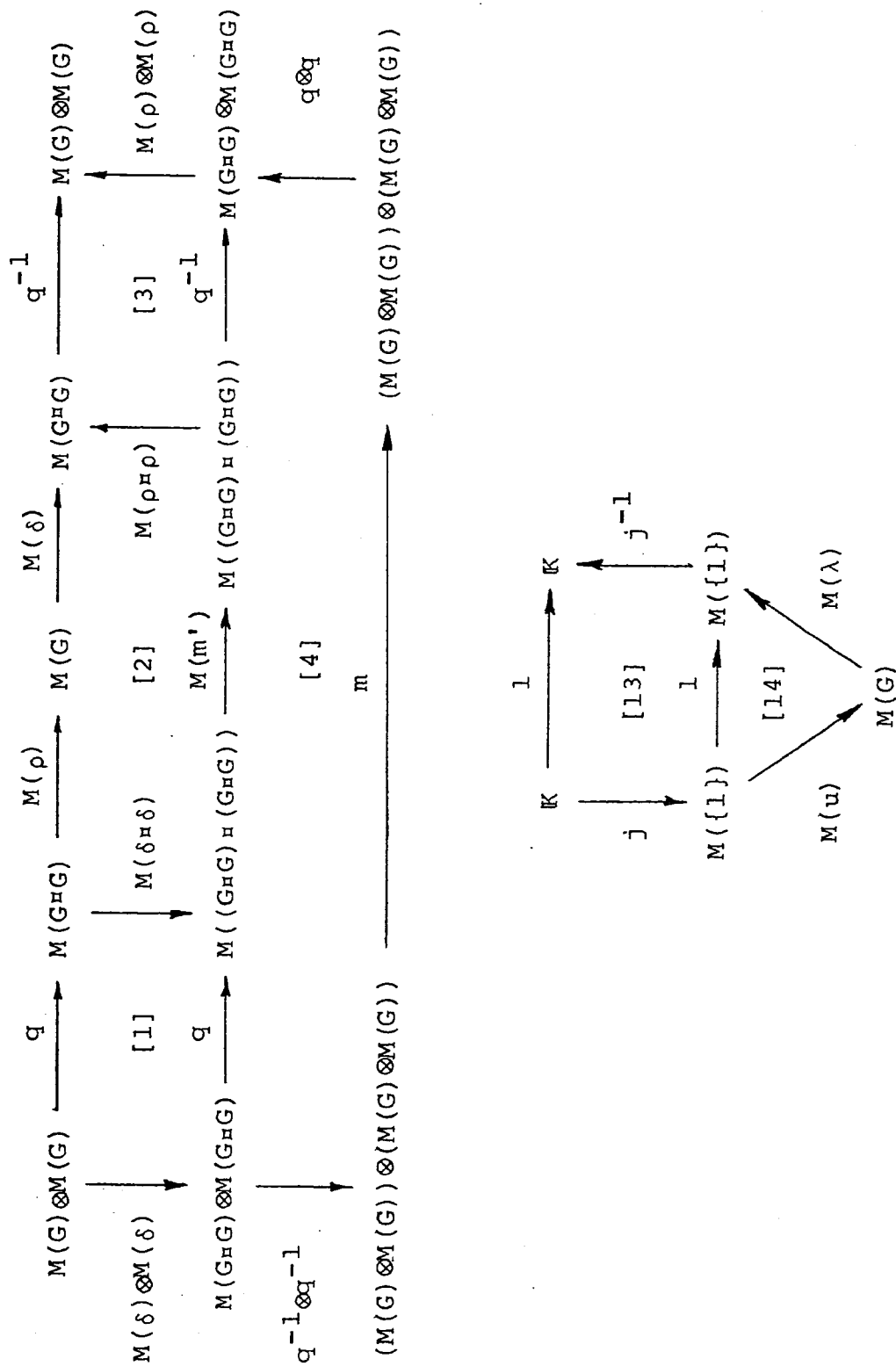
$(M(G), M(\rho) \circ q, M(u) \circ j)$ is a \mathcal{D} -algebra, provided G is a k -monoid.

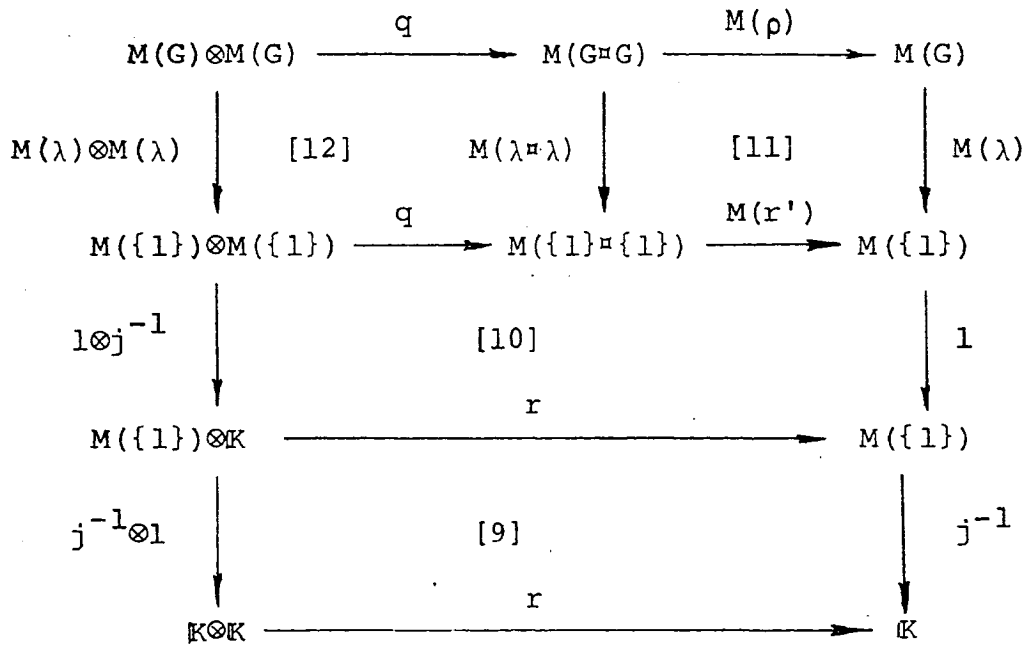
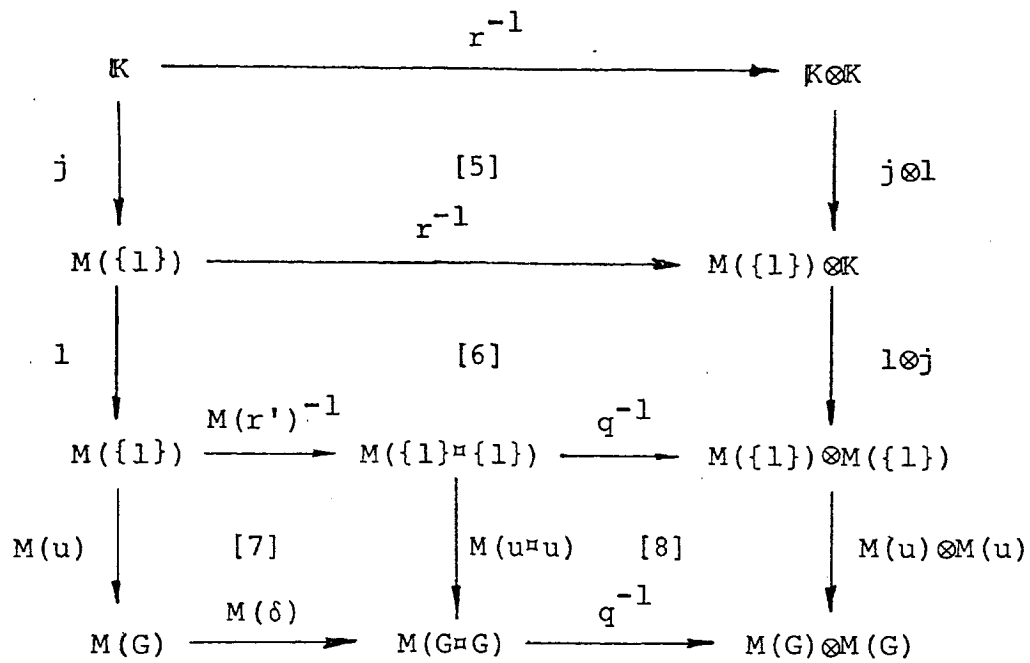
iii) Continue to assume that G is a k -monoid with identity e . Let $\rho, u, \delta,$ and λ be as in ii). Recall that m' and m denote the middle-four exchange of \mathcal{J} and \mathcal{D} respectively.

The following four diagrams commute:



Consider the following four large diagrams, each composed of various "subdiagrams".

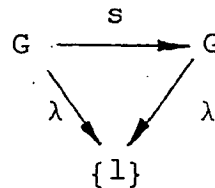
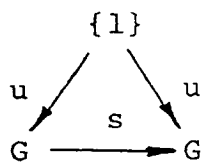
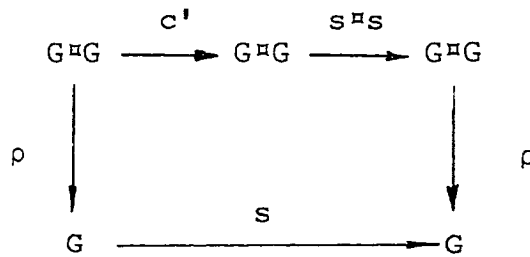
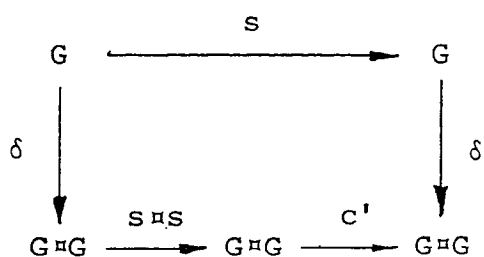
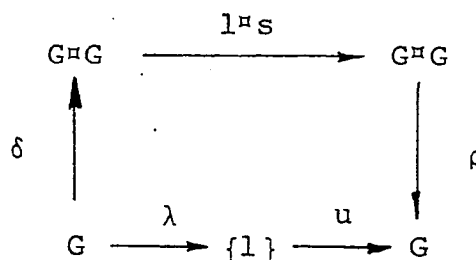
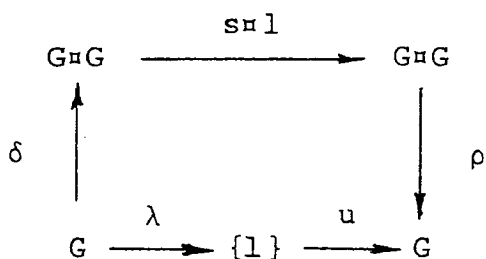




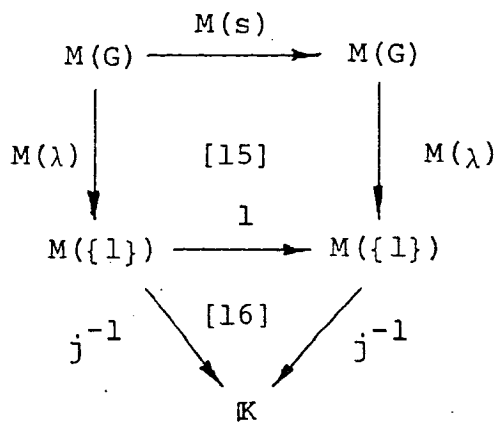
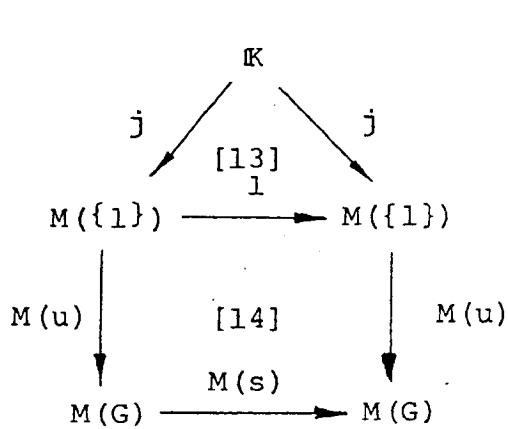
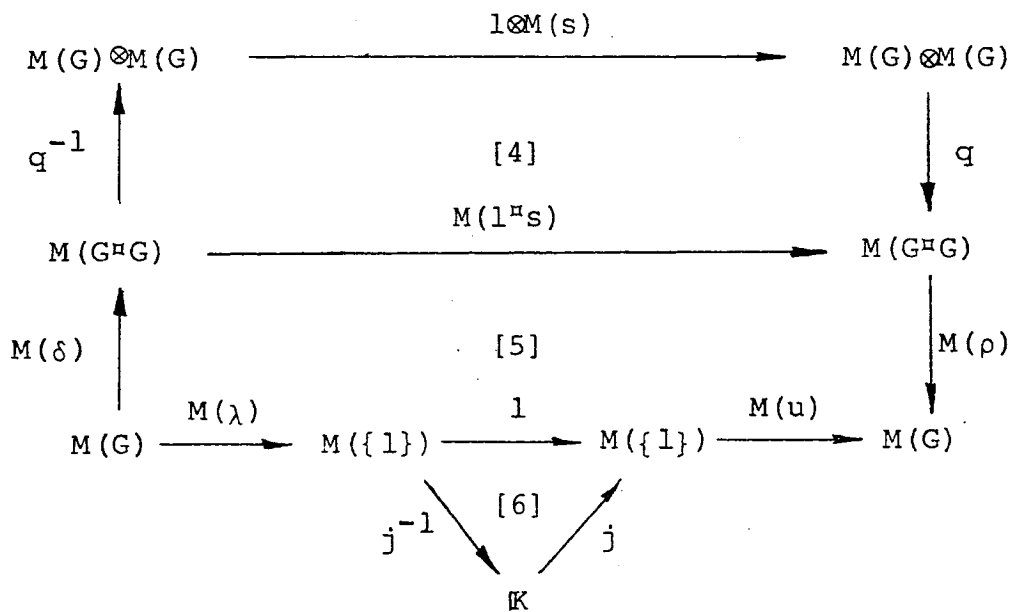
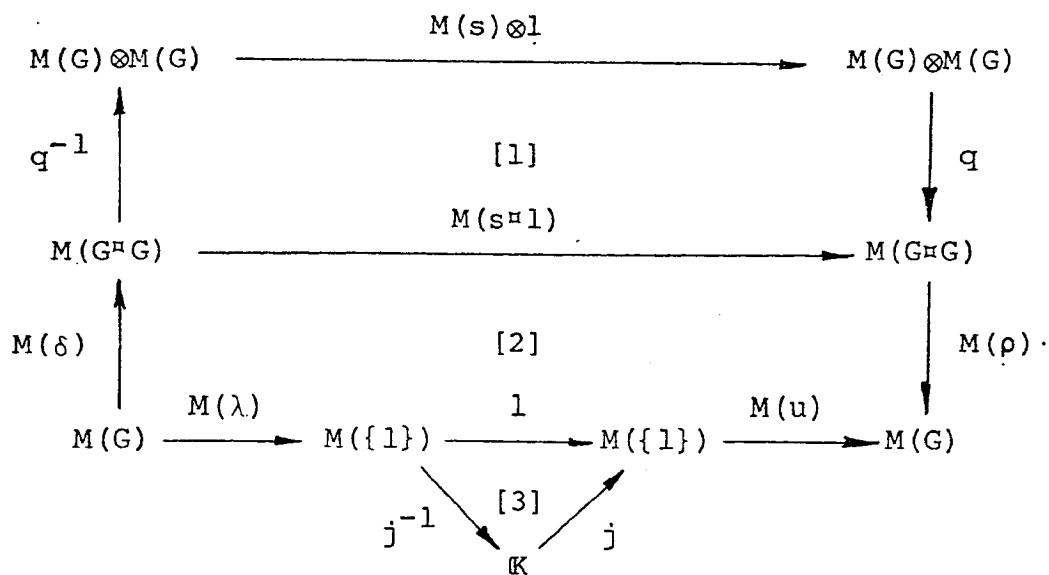
Diagrams [2], [7], [11], and [14] commute because the first group of four diagrams commute and because M is a functor. Diagrams [1], [3], [8], and [12] commute because q is natural. Diagram [4] commutes by 23.21. Diagrams [5] and [9] commute because r is natural.

Diagrams [6] and [10] commute because (M, q, j) is a strong symmetric functor. Diagram [13] commutes trivially. Therefore each of the four large diagrams (consisting of the outside arrows) commutes. This proves that $(M(G), M(\rho) \circ q, M(u) \circ j, q^{-1} \circ M(\delta), j^{-1} \circ M(\lambda))$ is a cocommutative \mathcal{D} -bialgebra, provided G is a k -monoid.

iv) Now suppose that G is a k -group. Let $\rho, u, \delta,$ and λ be as in ii). Define $s: G \rightarrow G$ by $s(x) = x^{-1}$. Then the following six diagrams commute:



Now consider the following six large diagrams:



$$\begin{array}{ccccc}
 M(G) \otimes M(G) & \xrightarrow{c} & M(G) \otimes M(G) & \xrightarrow{M(s) \otimes M(s)} & M(G) \otimes M(G) \\
 \downarrow q & & \downarrow q & & \downarrow q \\
 M(G \# G) & \xrightarrow{M(c')} & M(G \# G) & \xrightarrow{M(s \# s)} & M(G \# G) \\
 \downarrow M(\rho) & & \downarrow M(\rho) & & \downarrow M(\rho) \\
 M(G) & \xrightarrow{M(s)} & M(G) & & M(G)
 \end{array}$$

$$\begin{array}{ccccc}
 M(G) & \xrightarrow{M(s)} & M(G) & & M(G) \\
 \downarrow M(\delta) & & \downarrow M(\delta) & & \downarrow M(\delta) \\
 M(G \# G) & \xrightarrow{M(s \# s)} & M(G \# G) & \xrightarrow{M(c')} & M(G \# G) \\
 \downarrow q^{-1} & & \downarrow q^{-1} & & \downarrow q^{-1} \\
 M(G) \otimes M(G) & \xrightarrow{M(s) \otimes M(s)} & M(G) \otimes M(G) & \xrightarrow{c} & M(G) \otimes M(G)
 \end{array}$$

Diagrams [1], [4], [8], and [11] commute because q is natural. Diagrams [2], [5], [9], [10], [14], and [15] commute because M is a functor and since the first group of six diagrams commute. Diagrams [3], [6], [13], and [16] commute trivially. Diagrams [7] and [12] commute because (M, q, j) is a strong symmetric monoidal functor. Therefore each of the six large diagrams (consisting of the outside arrows) commute. Hence $(M(G), M(\rho) \circ q, M(u) \circ j, q^{-1} \circ M(\delta), j^{-1} \circ M(\lambda), M(s))$ is a cocommutative \mathcal{D} -Hopf algebra provided G is a k -group.

We are almost done showing that the objects of \mathcal{J} , \mathcal{S} , and \mathcal{R} are sent to objects of \mathcal{N} , \mathcal{M} , and \mathcal{L} . The only thing we have yet to show is that if G is a k -monoid or k -group, then $M(G)$ is a non-zero vector space. To this end

24.24a Lemma. If X is a k -space, $\varepsilon: X \rightarrow M(X)$ is the canonical map, and $x \in X$, then $\varepsilon(x) \neq 0$.

Proof of 24.24a: Suppose $x \in X$. Let $f: X \rightarrow K$ be defined by $f(t) = 1$ for all $t \in X$. f is continuous, so there exists a continuous linear map \bar{f} from $M(X)$ to K such that $\bar{f} \circ \varepsilon = f$. Hence $\bar{f}(\varepsilon(x)) = f(x) = 1$. Thus $\varepsilon(x) \neq 0$ due to the linearity of \bar{f} . QED on lemma

Noting that k -groups and k -monoids are non-empty, we see that if G is a k -group or k -monoid, then $M(G) \neq \{0\}$. Thus the functor M behaves properly with respect to the objects of the various categories.

Now let us look at the morphisms.

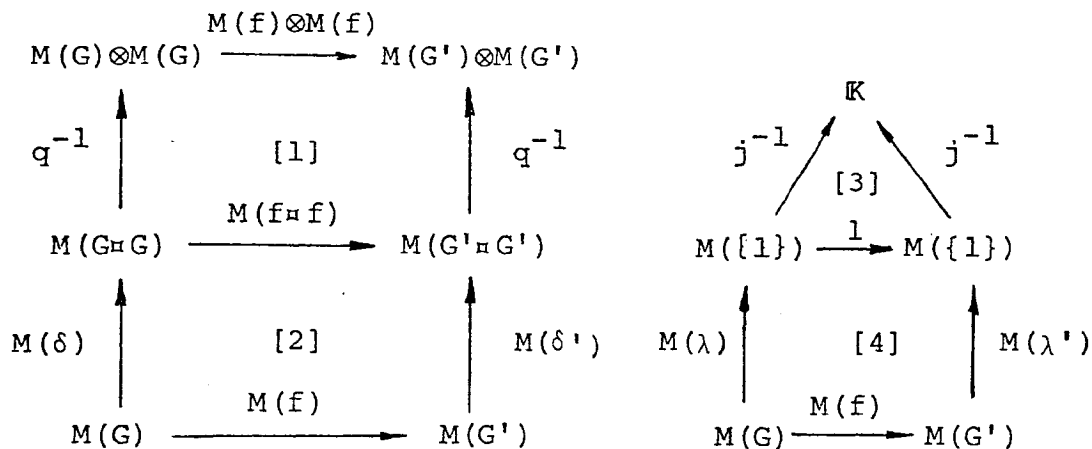
v) Suppose G and G' are k -spaces and $f: G \rightarrow G'$ is a continuous map.

Define $\delta: G \rightarrow G \sqcup G$ and $\delta': G' \rightarrow G' \sqcup G'$ by $x \mapsto (x, x)$; and $\lambda: G \rightarrow \{1\}$ and $\lambda': G' \rightarrow \{1\}$ by $x \mapsto 1$. Then the following two diagrams commute:

$$\begin{array}{ccc}
 G & \xrightarrow{f} & G' \\
 \delta \downarrow & & \downarrow \delta' \\
 G \sqcup G & \xrightarrow{f \sqcup f} & G' \sqcup G'
 \end{array}$$

$$\begin{array}{ccc}
 G & \xrightarrow{f} & G' \\
 \lambda \searrow & & \swarrow \lambda' \\
 & \{1\} &
 \end{array}$$

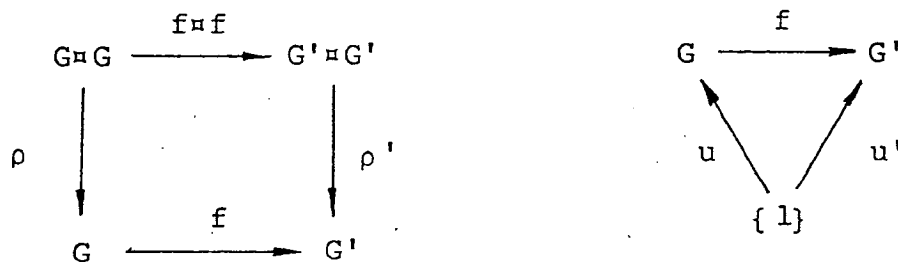
Consider the following two large diagrams:



Diagrams [2] and [4] commute since M is a functor and the first group of two diagrams commute. Diagrams [1] and [3] commute since (M, q, j) is a strong symmetric monoidal functor. Thus the two large diagrams (consisting of the outside arrows) commute. Hence $M(f)$ is a \mathcal{D} -coalgebra morphism, provided f is a continuous map.

vi) Now assume G and G' are k -monoids with identities e and e' respectively; and assume $f: G \rightarrow G'$ is a k -monoid morphism. Let $\rho: G \boxplus G \rightarrow G$ and $\rho': G' \boxplus G' \rightarrow G'$ be the multiplication maps of G and G' respectively. Let $u: \{1\} \rightarrow G$ and $u': \{1\} \rightarrow G'$ be defined by $u(1) = e$ and $u'(1) = e'$.

Then the following two diagrams commute:



By essentially a dual argument to that of v), one can show that $M(f)$ is a \mathcal{D} -algebra morphism. But then due to the definition of \mathcal{D} -bialgebra morphism, $M(f)$ is one of these too.

vii) Now assume that G and G' are k -groups, and assume that $f: G \rightarrow G'$ is a k -group morphism. Let $s: G \rightarrow G$ and $s': G' \rightarrow G'$ be defined by $x \mapsto x^{-1}$. Then the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{f} & G' \\ s \downarrow & & \downarrow s' \\ G & \xrightarrow{f} & G' \end{array}$$

Consider:

$$\begin{array}{ccc} M(G) & \xrightarrow{M(f)} & M(G') \\ M(s) \downarrow & & \downarrow M(s') \\ M(G) & \xrightarrow{M(f)} & M(G') \end{array}$$

It commutes since M is a functor and the diagram immediately above it commutes. Hence $M(f)$ is a \mathcal{D} -Hopf algebra morphism.

So taken together we will have shown that (M, q, j) is a functor from \mathcal{J} to \mathcal{N} , from \mathcal{S} to \mathcal{M} , and from \mathcal{R} to \mathcal{L} ; provided we can show that M behaves properly with respect to composition of morphisms and identity maps. But this follows since $M: \mathcal{J} \rightarrow \mathcal{D}$ is a functor.

QED on 24.24

Section 25 - An adjoint to $M: \mathcal{K} \rightarrow \mathcal{L}$.

In this section, in analogy to the strictly algebraic theory of coalgebras, we shall introduce a functor Ω , from \mathcal{D} -coalgebras to k -spaces, which will assign to every \mathcal{D} -coalgebra its "group-like elements". In a natural way, Ω can be shown to be a functor from \mathcal{D} -bialgebras to k -monoids and from \mathcal{D} -Hopf algebras to k -groups.

In each of the three cases it is shown that in a natural way, the canonical maps $\varepsilon = \varepsilon_X: X \rightarrow M(X)$ can be regarded to be a natural transformation which maps X to $\Omega(M(X))$. In addition if X is a topologically p -complete k -space, then ε_X will be an isomorphism in the appropriate category.

Finally, it is shown that in each of the three categorical situations, Ω is a coadjoint of M ; that is the functors $\text{Mor}_{\mathcal{Y}}(M(-), -)$ and $\text{Mor}_{\mathcal{Y}}(-, \Omega(-))$ are naturally isomorphic, provided $\mathcal{F} = \mathcal{D}$ -coalgebras and $\mathcal{G} = k$ -spaces; $\mathcal{F} = \mathcal{D}$ -bialgebras and $\mathcal{G} = k$ -monoids; or $\mathcal{F} = \mathcal{D}$ -Hopf algebras and $\mathcal{G} = k$ -groups.

25.1 Definition. If $\mathcal{C} = (C, \Delta, \zeta)$ is a \mathcal{D} -coalgebra, then $\{c \in C : \Delta(c) = c \otimes c \text{ and } c \neq 0\}$ is called the set of group-like elements of \mathcal{C} ; and it is denoted by $\Omega(\mathcal{C})$.

If \mathcal{A} is a \mathcal{D} -bialgebra or a \mathcal{D} -Hopf algebra, then

the group-like elements of \mathcal{O} will be defined to be equal to the group-like elements of the underlying \mathcal{D} -coalgebra of \mathcal{O} ; and the group-like elements of \mathcal{O} will be denoted by $\Omega(\mathcal{O})$.

25.2 Proposition. If $\mathcal{H} = (H, P, U, \Delta, \zeta)$ is a non-zero \mathcal{D} -bialgebra, then the group-like elements of \mathcal{H} together with the map $(x, y) \mapsto P(x \otimes y)$ form a monoid.

25.3 Proposition. If $\mathcal{H} = (H, P, U, \Delta, \zeta, S)$ is a non-zero \mathcal{D} -Hopf algebra, then the group-like elements of \mathcal{H} together with the map $(x, y) \mapsto P(x \otimes y)$ form a group.

25.4 Definition. If \mathcal{H} is a non-zero \mathcal{D} -bialgebra, let $\Omega(\mathcal{H})$ denote the monoid described above and call $\Omega(\mathcal{H})$ the characteristic monoid of \mathcal{H} .

If \mathcal{H} is a non-zero \mathcal{D} -Hopf algebra, let $\Omega(\mathcal{H})$ denote the group described above and call $\Omega(\mathcal{H})$ the characteristic group of \mathcal{H} .

Proof of 25.2 and 25.3: First suppose that \mathcal{H} is a non-zero \mathcal{D} -bialgebra. Suppose $x, y \in \Omega(\mathcal{H})$.

Claim: $P(x \otimes y) \in \Omega(\mathcal{H})$.

Proof of claim: Since Δ is an algebra morphism [i.e. diagram a) of 24.5 commutes], and since $\Delta(x) = x \otimes x$ and $\Delta(y) = y \otimes y$; we see that $\Delta(P(x \otimes y)) = P(x \otimes y) \otimes P(x \otimes y)$. Now suppose $P(x \otimes y) = 0$. Now since ζ is an algebra morphism [i.e. diagram c) of 24.5 commutes], $\zeta(x)\zeta(y) = 0$. So either $\zeta(x) = 0$ or $\zeta(y) = 0$. But consider the following lemma.

25.3a Lemma. If $\mathcal{H} = (H, \Delta, \zeta)$ is a \mathcal{D} -coalgebra and $x \in \Omega(\mathcal{H})$, then $\zeta(x) = 1$.

Proof of 25.3a: ζ is a left counit [i.e. diagram b) of 24.3 commutes]. So if $\Delta(x) = x \otimes x$, then $\zeta(x)x = x$. But if $x \neq 0$, this implies that $\zeta(x) = 1$. QED on 25.3a

Thus we have a contradiction. Hence $P(x \otimes y) \neq 0$.
So $P(x \otimes y) \in \Omega(\mathcal{H})$. QED on claim

The proposed monoid multiplication is associative since P is associative [i.e. diagram a) of 24.1 commutes].

Claim: $U(1) \in \Omega(\mathcal{H})$.

Proof of claim: $\Delta(U(1)) = U(1) \otimes U(1)$, because U is a coalgebra morphism [i.e. diagram b) of 24.5 commutes], and $U(1) \neq 0$ by the following lemma.

25.3b Lemma. If $\mathcal{H} = (H, P, U)$ is a non-zero \mathcal{D} -algebra, then $U(1) \neq 0$.

Proof of 25.3b: U is a right unit [i.e. diagram c) of 24.1 commutes]. So if $U(1) = 0$, then $H = \{0\}$.
Contradiction. QED on 25.3b

QED on claim

Also $U(1)$ acts as an identity on $\Omega(\mathcal{H})$, since U is a unit for \mathcal{H} [i.e. diagram b) and c) of 24.1 commute].

Thus we have that $\Omega(\mathcal{H})$ is a monoid if \mathcal{H} is a non-zero \mathcal{D} -bialgebra.

Now suppose \mathcal{H} is a \mathcal{D} -Hopf algebra with antipode $S: H \rightarrow H$.

Then if $x \in \Omega(\mathcal{H})$, we have $\Delta(S(x)) = S(x) \otimes S(x)$ because S is an anti-algebra morphism [i.e. diagram c)

of 24.6 commutes].

Claim: If $x \in \Omega(\mathcal{H})$, then $S(x) \in \Omega(\mathcal{H})$ and $S(x)$ is the inverse of x in $\Omega(\mathcal{H})$.

Proof of claim: Let $x \in \Omega(\mathcal{H})$. Then $\zeta(x) = 1$ by 25.3a. So since diagrams a) and b) of 24.6 commute, we have $P(S(x) \otimes x) = U(1) = P(x \otimes S(x))$. Now if $S(x) = 0$, then $U(1) = 0$. But this is not the case by 25.3b. So $S(x) \neq 0$. Thus $S(x) \in \Omega(\mathcal{H})$. Hence the above equations tell us that $S(x) = x^{-1}$. QED on claim

Hence $\Omega(\mathcal{H})$ with the multiplication map $(x, y) \mapsto P(x \otimes y)$ is a group provided \mathcal{H} is a non-zero \mathcal{D} -Hopf algebra. QED on 25.2 & 25.3

25.5 Proposition. Let \mathcal{H} be a non-zero \mathcal{D} -Hopf algebra (resp., non-zero \mathcal{D} -bialgebra). Endow $\Omega(\mathcal{H})$ with the topology which is the coreflection in the category of k -spaces of the relative topology on $\Omega(\mathcal{H})$ (cf. C3). Then $\Omega(\mathcal{H})$ is a k -group (resp., k -monoid).

Proof: Let $\mathcal{H} = (H, P, U, \Delta, \zeta)$ be a non-zero \mathcal{D} -bialgebra. For the sake of clarity, let \mathcal{J}_0 denote the relative topology on $\Omega(\mathcal{H})$ induced by H and let k denote the coreflection functor from the category of Hausdorff topological spaces to the category of k -spaces. Thus the topology with which we just endowed $\Omega(\mathcal{H})$ is just $k((\Omega(\mathcal{H}), \mathcal{J}_0))$.

Let R be a compact subset of $(\Omega(\mathcal{H}), \mathcal{J}_0) \times (\Omega(\mathcal{H}), \mathcal{J}_0)$, where \times denotes the product topology. The map from $H \times H \rightarrow H$ defined by $(x, y) \mapsto P(x \otimes y)$ is hypocontin-

uous. Hence by prop. 5 of §4 of chap. 3 of [4], this map restricted to R is continuous. Thus the map $v: (\Omega(\mathcal{H}), \mathcal{J}_0) \times (\Omega(\mathcal{H}), \mathcal{J}_0) \rightarrow (\Omega(\mathcal{H}), \mathcal{J}_0)$ defined by $v(x, y) = P(x \otimes y)$, has the property that when restricted to compact sets it is continuous. So by theorem 3.2 of [45], $k(v): k((\Omega(\mathcal{H}), \mathcal{J}_0) \times (\Omega(\mathcal{H}), \mathcal{J}_0)) \rightarrow k((\Omega(\mathcal{H}), \mathcal{J}_0))$ is continuous. But by lemma 4.5 of [45], $k((\Omega(\mathcal{H}), \mathcal{J}_0) \times (\Omega(\mathcal{H}), \mathcal{J}_0)) = k((\Omega(\mathcal{H}), \mathcal{J}_0)) \square k((\Omega(\mathcal{H}), \mathcal{J}_0))$. Hence the multiplication map is continuous for the proper topologies. Hence $\Omega(\mathcal{H})$ with the topology $k((\Omega(\mathcal{H}), \mathcal{J}_0))$ is a k -monoid.

Now assume that \mathcal{H} is a non-zero \mathcal{D} -Hopf algebra. Let $S: \mathcal{H} \rightarrow \mathcal{H}$ be the antipode. S is continuous, so $S|_{\Omega(\mathcal{H})}: (\Omega(\mathcal{H}), \mathcal{J}_0) \rightarrow (\Omega(\mathcal{H}), \mathcal{J}_0)$ is continuous. Thus $k(S|_{\Omega(\mathcal{H})}): k((\Omega(\mathcal{H}), \mathcal{J}_0)) \rightarrow k((\Omega(\mathcal{H}), \mathcal{J}_0))$ is continuous. But $S|_{\Omega(\mathcal{H})}$ is the inverse map of the group $\Omega(\mathcal{H})$. Hence $\Omega(\mathcal{H})$ with the topology $k((\Omega(\mathcal{H}), \mathcal{J}_0))$ is a k -group. QED

25.6 Convention. Let \mathcal{O} be a \mathcal{D} -coalgebra, \mathcal{D} -bialgebra, or \mathcal{D} -Hopf algebra. When not mentioned otherwise $\Omega(\mathcal{O})$ will be endowed with the topology which is the coreflection in the category of k -spaces of the relative topology on $\Omega(\mathcal{O})$ induced by the underlying topological vector space of \mathcal{O} .

Let $\eta, \mathcal{M}, \mathcal{L}, \mathcal{J}, \mathcal{S}$, and \mathcal{R} be as in 24.23 and 10.1.

25.7 Theorem. Ω can be extended to the morphisms of \mathcal{N} , \mathcal{M} , and \mathcal{L} as follows: if $f: A \rightarrow B$ is a morphism, then $\Omega(f) = f|_{\Omega(A)}$.

Thus extended, Ω is a functor from \mathcal{N} to \mathcal{J} , from \mathcal{M} to \mathcal{S} , and from \mathcal{L} to \mathcal{R} .

Proof: Let us first restrict our attention to \mathcal{D} -coalgebras and k -spaces. If \mathcal{H} is a \mathcal{D} -coalgebra, then $\Omega(\mathcal{H})$ is obviously a k -space (cf. 25.6).

Suppose $\mathcal{H} = (H, \Delta, \zeta)$ and $\mathcal{H}' = (H', \Delta', \zeta')$ are \mathcal{D} -coalgebras and $f: \mathcal{H} \rightarrow \mathcal{H}'$ is a \mathcal{D} -coalgebra morphism.

Claim: If $x \in \Omega(\mathcal{H})$, then $f(x) \in \Omega(\mathcal{H}')$.

Proof of claim: It follows from the fact that f preserves comultiplication [cf. diagram a) of 24.10], that if $x \in \Omega(\mathcal{H})$, then

$$\Delta'(f(x)) = (f \otimes f)(\Delta(x)) = (f \otimes f)(x \otimes x) = f(x) \otimes f(x). \text{ Since}$$

f also preserves counits [cf. diagram b) of 24.10],

$$\zeta(x) = \zeta'(f(x)). \text{ But by 25.3a, } \zeta(x) = 1. \text{ Hence}$$

$$f(x) \neq 0. \text{ So } f(x) \in \Omega(\mathcal{H}'). \quad \text{QED on claim}$$

Now since f is continuous, if we give the relative topology to both $\Omega(\mathcal{H})$ and $\Omega(\mathcal{H}')$, then

$f|_{\Omega(\mathcal{H})}: \Omega(\mathcal{H}) \rightarrow \Omega(\mathcal{H}')$ will be continuous. But now using coreflection properties, we have that

$f|_{\Omega(\mathcal{H})}: \Omega(\mathcal{H}) \rightarrow \Omega(\mathcal{H}')$ is continuous if $\Omega(\mathcal{H})$ and $\Omega(\mathcal{H}')$ are given their k -space topologies (cf. 25.6).

So we have that Ω maps $\text{Mor}_{\mathcal{N}}(\mathcal{H}, \mathcal{H}')$ into $\text{Mor}_{\mathcal{J}}(\Omega(\mathcal{H}), \Omega(\mathcal{H}'))$. That Ω maps identity maps to identity maps and that composition behaves properly with respect to Ω is trivial. Hence Ω is a functor from

\mathcal{M} to \mathcal{S} .

Now let us consider non-zero \mathcal{D} -bialgebras and k -monoids. We have already shown in 25.5 that if \mathcal{A} is a non-zero \mathcal{D} -bialgebra, then $\Omega(\mathcal{A})$ is a k -monoid. Using the fact that $\Omega: \mathcal{M} \rightarrow \mathcal{S}$ is a functor; in order to show that $\Omega: \mathcal{M} \rightarrow \mathcal{S}$ is a functor, it is only necessary to prove that if $\mathcal{A} = (H, P, U, \Delta, \zeta)$ and $\mathcal{A}' = (H', P', U', \Delta', \zeta')$ are non-zero \mathcal{D} -bialgebras and $f: \mathcal{A} \rightarrow \mathcal{A}'$ is a \mathcal{D} -bialgebra morphism, then $\Omega(f) = f|_{\Omega(\mathcal{A})}$ is a semigroup homomorphism which preserves identities.

Since f preserves units [cf. diagram b) of 24.9], we have $f(U(1)) = U'(1)$. But $U(1)$ and $U'(1)$ are the identities of $\Omega(\mathcal{A})$ and $\Omega(\mathcal{A}')$ respectively. So $\Omega(f)$ preserves identities. f preserves the multiplication of \mathcal{A} and \mathcal{A}' [cf. diagram a) of 24.9]; so recalling that $(x, y) \mapsto P(x \otimes y)$ and $(r, s) \mapsto P'(r \otimes s)$ are the multiplication maps of $\Omega(\mathcal{A})$ and $\Omega(\mathcal{A}')$ respectively, we see that $\Omega(f) = f|_{\Omega(\mathcal{A})}$ is a semigroup homomorphism.

So Ω maps $\text{Mor}_{\mathcal{M}}(\mathcal{A}, \mathcal{A}')$ into $\text{Mor}_{\mathcal{S}}(\Omega(\mathcal{A}), \Omega(\mathcal{A}'))$. Hence $\Omega: \mathcal{M} \rightarrow \mathcal{S}$ is a functor.

Finally let us consider non-zero \mathcal{D} -Hopf algebras and k -groups.

If \mathcal{A} is a non-zero \mathcal{D} -Hopf algebra, then $\Omega(\mathcal{A})$ is a k -group by 25.5. So $\Omega: \mathcal{L} \rightarrow \mathcal{R}$ is a functor since $\Omega: \mathcal{M} \rightarrow \mathcal{S}$ is a functor. This is true because group homomorphisms are simply semigroup homomorphisms. QED

25.8 Proposition. If G is a k -group (k -monoid, or k -space respectively) and if $\varepsilon: G \rightarrow M(G)$ is the canonical map, then as sets $\varepsilon(G) \subset \Omega(M(G))$, and $\varepsilon: G \rightarrow \Omega(M(G))$ is a k -group (k -monoid, or k -space respectively) morphism.

Proof: Assume initially that G is only a k -space. Let us begin by proving the following

25.8a Lemma. $\varepsilon(G) \subset \Omega(M(G))$.

Proof of lemma: Let $\delta: G \rightarrow G \sharp G$ be the diagonal map. Let $q: M(G) \otimes M(G) \rightarrow M(G \sharp G)$ be the natural isomorphism. The comultiplication of the \mathcal{D} -coalgebra $M(G)$ is just $q^{-1} \circ M(\delta)$. Consider

$$\begin{array}{ccccc}
 G & \xrightarrow{\delta} & G \sharp G & & \\
 \varepsilon \downarrow & & \varepsilon \downarrow & \searrow r & \\
 & [1] & & & M(G) \otimes M(G) \\
 M(G) & \xrightarrow{M(\delta)} & M(G \sharp G) & \nearrow q^{-1} & \\
 & & & &
 \end{array}$$

where r is as in 23.11. Diagram [2] commutes by 23.11 and diagram [1] commutes by the definition of $M(\delta)$. Hence the large diagram formed by the outside arrows commutes.

Thus for all $x \in G$, $(q^{-1} \circ M(\delta))(\varepsilon(x)) = \varepsilon(x) \otimes \varepsilon(x)$. But also if $x \in G$, then $\varepsilon(x) \neq 0$ by 24.24a. Hence for all $x \in G$, $\varepsilon(x) \in \Omega(M(G))$. QED on lemma

Thus since $\varepsilon: G \rightarrow M(G)$ is continuous, $\varepsilon: G \rightarrow \Omega(M(G))$ is continuous provided $\Omega(M(G))$ has the relative topology induced by $M(G)$. But since G is a

k -space, ε is continuous from G to the coreflection in the category of k -spaces of $\Omega(M(G))$, i.e. ε is continuous from G to $\Omega(M(G))$ where the latter space has its k -space topology.

Now assume G is a k -monoid with identity e . Let $\rho: G \times G \rightarrow G$ denote the multiplication map. Again let $q: M(G) \otimes M(G) \rightarrow M(G \times G)$ be the canonical natural isomorphism. The multiplication of the \mathcal{D} -bialgebra $M(G)$ is then just $M(\rho) \circ q$.

Consider

$$\begin{array}{ccccc}
 & & G \times G & \xrightarrow{\rho} & G \\
 & \swarrow r & \downarrow & \varepsilon & \downarrow \varepsilon \\
 M(G) \otimes M(G) & [1] & & [2] & \\
 & \searrow q & M(G \times G) & \xrightarrow{M(\rho)} & M(G)
 \end{array}$$

where r is as in 23.11. Diagram [1] commutes by 23.11 and diagram [2] commutes by the definition of $M(\rho)$. So the diagram consisting of the outside arrows commutes. Hence for all $x, y \in G$, $(M(\rho) \circ q)(\varepsilon(x) \otimes \varepsilon(y)) = \varepsilon(\rho(x, y))$. Hence I will have shown that ε is a k -monoid morphism if I can show that ε preserves identities.

Define $u: \{1\} \rightarrow G$ by $u(1) = e$. Then the diagram

$$\begin{array}{ccc}
 \{1\} & \xrightarrow{\varepsilon} & M(\{1\}) \\
 u \downarrow & & \downarrow M(u) \\
 G & \xrightarrow{\varepsilon} & M(G)
 \end{array}$$

commutes due to the definition of $M(u)$. Now $M(u) \circ j$

is the unit of the \mathcal{D} -bialgebra $M(G)$, where
 $j: \mathbb{K} \rightarrow M(\{1\})$ is the canonical isomorphism (cf. 23.12).
Hence the identity of the k -monoid $\Omega(M(G))$ is
 $(M(u) \circ j)(1) = (M(u))(j(1)) = M(u)(\varepsilon(1)) = (M(u) \circ \varepsilon)(1) =$
 $\varepsilon(u(1)) = \varepsilon(e)$. So $(M(u) \circ j)(1) = \varepsilon(e)$. Thus ε
sends the identity of G to the identity of $\Omega(M(G))$.
Hence ε is a k -monoid morphism.

Assume G is a k -group. Then G is also a k -monoid. So $\varepsilon: G \rightarrow \Omega(M(G))$ is a semigroup homomorphism. But semigroup homomorphisms between groups are group homomorphisms. Hence $\varepsilon: G \rightarrow \Omega(M(G))$ is a k -group morphism. QED

25.9 Theorem. Let X be a topologically p -complete k -space. Let $\varepsilon: X \rightarrow \Omega(M(X))$ be the canonical map. Then

- 1) ε is a k -space isomorphism;
- 2) if X is a k -monoid, then ε is a k -monoid isomorphism; and
- 3) if X is a k -group, then ε is a k -group isomorphism.

Proof: Let us initially assume only that X is a topologically p -complete k -space. Recall that by 14.1, $M(X) = [c_c(X)]^P$. We will begin by proving two lemmas.

25.9a Lemma. If we consider $c_c(X)$ as being an algebra whose multiplication is defined pointwise, then $\Omega(M(X))$ equals the set of all unit preserving continuous algebra homomorphisms from $c_c(X)$ to \mathbb{K} .

Proof: Let $\delta: X \rightarrow X \times X$ be the diagonal map, $\mu \in \mathcal{M}(X)$ and $f, g \in c(X)$. Let juxtaposition denote the product in $c_c(X)$. Thus $(fg)(x) = f(x)g(x)$ for all $x \in X$. Consider the map $h: X \times X \rightarrow \mathbb{K}$ defined by $h(x, y) = f(x)g(y)$. Note that $fg = h \circ \delta$. By 23.10, $q(\mu \otimes \mu)(h) = \mu(y \mapsto \mu(x \mapsto h(x, y)))$. Fix y for a moment and consider the function $x \mapsto h(x, y) = f(x)g(y)$. It is simply the function $g(y)f$. So $\mu(x \mapsto h(x, y)) = \mu(g(y)f) = g(y)\mu(f)$. Hence the function $y \mapsto \mu(x \mapsto h(x, y))$ is just $y \mapsto \mu(f)g(y)$; that is $\mu(f)g$. So $\mu(y \mapsto \mu(x \mapsto h(x, y))) = \mu(\mu(f)g) = \mu(f)\mu(g)$. So $[q(\mu \otimes \mu)](h) = \mu(f)\mu(g)$.

Also note that $[[M(\delta)](\mu)](h) = \mu(h \circ \delta) = \mu(fg)$.

Now assume $\mu \in \Omega(\mathcal{M}(X))$, i.e. $[q^{-1} \circ M(\delta)](\mu) = \mu \otimes \mu$ and $\mu \neq 0$. Then $[M(\delta)](\mu) = q(\mu \otimes \mu)$. So by the above $\mu(fg) = [[M(\delta)](\mu)](h) = q(\mu \otimes \mu)(h) = \mu(f)\mu(g)$. Hence for all $f, g \in c(X)$, $\mu(fg) = \mu(f)\mu(g)$. Let $\mathbb{1}$ denote the constantly one function. Then $\mu(\mathbb{1}) = \mu(\mathbb{1}\mathbb{1}) = \mu(\mathbb{1})\mu(\mathbb{1})$. So either $\mu(\mathbb{1}) = 0$ or $\mu(\mathbb{1}) = 1$. But if $\mu(\mathbb{1}) = 0$, then $\mu(f) = \mu(f\mathbb{1}) = \mu(f)\mu(\mathbb{1}) = 0$ for all $f \in c(X)$. But $\mu \neq 0$. Hence $\mu(\mathbb{1}) = 1$.

Hence we have shown that μ is a unit preserving continuous algebra homomorphism if we assume that $\mu \in \Omega(\mathcal{M}(X))$. So we are half done.

Let $\mu: c_c(X) \rightarrow \mathbb{K}$ be a continuous unit preserving algebra homomorphism. Let $f, g \in c(X)$. Define $h: X \times X \rightarrow \mathbb{K}$ by $h(x, y) = f(x)g(y)$. Since μ is an algebra morphism $\mu(fg) = \mu(f)\mu(g)$. So by early in the

first part of the proof, we find that

$([M(\delta)](\mu))(h) = \mu(fg) = \mu(f)\mu(g) = [q(\mu \otimes \mu)](h)$. But by 16.1 and 15.1, the linear span of functions such as h are dense in $c_c(X \times X)$. So if we knew that $M(\delta)(\mu)$ and $q(\mu \otimes \mu)$ were in $c_c(X \times X)'$, we could conclude that for all $r \in c(X \times X)$, $[M(\delta)(\mu)](r) = [q(\mu \otimes \mu)](r)$. To this end, let us prove the following two sublemmas.

25.9b Sublemma. If X and Y are k -spaces, $f: X \rightarrow Y$ is continuous, and X is topologically p -complete, then $\mu \in M(X)$ implies $M(f)(\mu) \in c_c(Y)' \subset M(Y)$.

Proof of 25.9b: The hypothesis implies that $\mu \in c_c(X)'$ (cf. 14.1). Suppose $g_\alpha \rightarrow 0$ in $c_c(Y)$. Then $[M(f)(\mu)](g_\alpha) = \mu(g_\alpha \circ f)$ by 10.8. Thus the continuity of f implies that $[M(f)(\mu)](g_\alpha) \rightarrow 0$. QED

25.9c Sublemma. If X and Y are topologically p -complete k -spaces, $\mu \in M(X)$, and $\nu \in M(Y)$, then $q(\mu \otimes \nu) \in c_c(X \times Y)' \subset M(X \times Y)$.

Proof of 25.9c: By 14.1, $\mu \in c_c(X)'$ and $\nu \in c_c(Y)'$. Let U be a neighborhood of zero in \mathbb{K} . Then there exists L compact in Y and a neighborhood V of zero in \mathbb{K} such that $g \in c(Y)$ and $g(L) \subset V$ implies $\nu(g) \in U$. Choose a neighborhood W of zero in \mathbb{K} and a compact subset J in X such that $f \in c(X)$ and $f(J) \subset W$ implies $\mu(f) \in V$.

Recall from 23.10 that if $h \in c(X \times Y)$, then $q(\mu \otimes \nu)(h) = \nu(y \mapsto \mu(x \mapsto h(x, y)))$. Thus if $h \in c(X \times Y)$ and $h(j, \ell) \in W$ for all $j \in J$ and $\ell \in L$, then $q(\mu \otimes \nu)(h) \in U$. Thus $q(\mu \otimes \nu) \in c_c(X \times Y)'$. QED on 25.9c

Using the last two sublemmas, we may conclude that $M(\delta)(\mu) = q(\mu \otimes \mu)$, i.e. $[q^{-1} \circ M(\delta)](\mu) = \mu \otimes \mu$. Now μ preserves units, so $\mu \neq 0$. Hence $\mu \in \Omega(M(X))$. Thus every continuous unit preserving algebra homomorphism from $c_c(X)$ to \mathbb{K} is in $\Omega(M(X))$. QED on 25.9a

25.9d Lemma. The set of unit preserving continuous algebra homomorphisms from $c_c(X)$ to \mathbb{K} are precisely the point evaluations of X , i.e. $\varepsilon(X)$.

Proof of 25.9d: The fact that point evaluations are continuous unit preserving algebra homomorphisms is easily verified.

So assume $\mu: c_c(X) \rightarrow \mathbb{K}$ is a continuous unit preserving algebra homomorphism. Let $K = \text{Supp } \mu$. Let $i: K \rightarrow X$ be the inclusion map. By 14.16, there exists a $v \in M(K)$ such that $[M(i)](v) = \mu$. I claim that v is a unit preserving continuous algebra homomorphism from $c_c(K)$ to \mathbb{K} .

Suppose $f, g \in c(K)$. Let $\sigma: c_c(K) \rightarrow c_c(X)$ be a continuous map such that $[\sigma(h)] \circ i = h$ for all $h \in c(K)$ (cf. 11.2). Then we have $v(h) = v(\sigma(h) \circ i) = M(i)(v)[\sigma(h)] = \mu(\sigma(h))$ for all $h \in c(K)$. So since $K = \text{Supp } \mu$ and since $[\sigma(fg)] \circ i = [\sigma(f)\sigma(g)] \circ i$, we have $v(fg) = \mu(\sigma(fg)) = \mu(\sigma(f)\sigma(g)) = \mu(\sigma(f))\mu(\sigma(g)) = v(f)v(g)$. Also let $\mathbb{1}_K$ and $\mathbb{1}_X$ denote the constantly one functions on K and X respectively. Then since $[\sigma(\mathbb{1}_K)] \circ i = \mathbb{1}_X \circ i$, we have $v(\mathbb{1}_K) = \mu(\sigma(\mathbb{1}_K)) = \mu(\mathbb{1}_X) = 1$. So v is a unit preserving continuous algebra

homomorphism from $c_c(K)$ to K .

But then by a classical theorem of functional analysis, there exists a $k \in K$ such that $v(f) = f(k)$ for all $f \in c(K)$. Now let $f \in c(X)$. Since $M(i)(v) = \mu$, we have

$$\mu(f) = [M(i)(v)](f) = v(f \circ i) = (f \circ i)(k) = f(i(k)) = f(k).$$

So for all $f \in c(X)$, $\mu(f) = f(k) = [\varepsilon(k)](f)$, i.e.

$$\mu = \varepsilon(k). \quad \text{QED on 25.9d}$$

So at this point we have, by use of 25.9a and 25.9d, that $\varepsilon(X) = \Omega(M(X))$. However by 14.10, ε is a homeomorphism onto its range. This implies that $\Omega(M(X))$ with the relative topology induced by $M(X)$ is a k -space. Hence $\Omega(M(X))$ with its k -space topology (cf. 25.6) equals $\Omega(M(X))$ with the relative topology induced by $M(X)$. Hence ε is a homeomorphism from X to $\Omega(M(X))$, i.e. ε is a k -space isomorphism.

Now if in addition we assume that X is a k -monoid, by 25.8 we know that $\varepsilon: X \rightarrow \Omega(M(X))$ is a k -monoid morphism. But bijective semigroup homomorphisms which preserve identities have the property that their inverses are again semigroup homomorphisms which preserve identities. Hence $\varepsilon^{-1}: \Omega(M(X)) \rightarrow X$ is a k -monoid morphism. So ε is a k -monoid isomorphism.

Finally, if in addition we assume that X is a k -group, then since k -groups are k -monoids and since k -monoid morphisms between k -groups are in fact k -group morphisms, we find that ε is a k -group isomorphism.

QED on 25.9

25.10 Theorem. Let $\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{R}, \mathcal{S}$, and \mathcal{J} be as in 24.23 and 10.1. Consider the functors each denoted by M from \mathcal{R} to \mathcal{L} , from \mathcal{S} to \mathcal{M} , and from \mathcal{J} to \mathcal{N} (cf. 24.24); and consider the functors each denoted by Ω from \mathcal{L} to \mathcal{R} , from \mathcal{M} to \mathcal{S} , and from \mathcal{N} to \mathcal{J} (cf. 25.7). Then in each of the three situations M is an adjoint of Ω ; or equivalently,

for all G in the appropriate category, $M(G)$ is in the appropriate category and there exists a morphism $\varepsilon_G: G \rightarrow \Omega(M(G))$ such that for all \mathcal{H} in the appropriate category if $f: G \rightarrow \Omega(\mathcal{H})$ is a morphism, then there exists a unique morphism $\bar{f}: M(G) \rightarrow \mathcal{H}$ such that the following diagram commutes:

$$\begin{array}{ccc}
 G & \xrightarrow{f} & \Omega(\mathcal{H}) \\
 \varepsilon_G \downarrow & \nearrow & \\
 \Omega(M(G)) & & \Omega(\bar{f})
 \end{array}$$

Proof: First of all the equivalence of the two forms of the conclusion is proved in 6.3.1 and 6.4.4 (1) of [18]. I shall prove the theorem in the context of the second form. I will begin the proof by proving a lemma.

25.10a Lemma. Let \mathcal{A} be a non-zero \mathcal{D} -Hopf algebra (non-zero \mathcal{D} -bialgebra, or \mathcal{D} -coalgebra respectively) with underlying locally convex space H . Let $i: \Omega(\mathcal{A}) \rightarrow H$ be the inclusion map. Let $\bar{i}: M(\Omega(\mathcal{A})) \rightarrow H$ be the unique continuous linear map which makes the following diagram commute:

$$\begin{array}{ccc}
 \Omega(\mathcal{H}) & \xrightarrow{i} & H \\
 \downarrow \varepsilon & \nearrow \bar{i} & \\
 M(\Omega(\mathcal{H})) & &
 \end{array}$$

Then \bar{i} is a \mathcal{D} -Hopf algebra morphism (\mathcal{D} -bialgebra morphism, or \mathcal{D} -coalgebra morphism respectively).

Proof: Note that $i: \Omega(\mathcal{H}) \rightarrow H$ is continuous since the topology of $\Omega(\mathcal{H})$ is finer than the relative topology induced by H (cf. 25.6).

Let $\mathcal{H} = (H, \Delta, \zeta)$ be a \mathcal{D} -coalgebra. Let $\lambda: \Omega(\mathcal{H}) \rightarrow \{1\}$ be the identically one map, $\delta: \Omega(\mathcal{H}) \rightarrow \Omega(\mathcal{H}) \sqcup \Omega(\mathcal{H})$ be the diagonal map and $q: M(\Omega(\mathcal{H})) \otimes M(\Omega(\mathcal{H})) \rightarrow M(\Omega(\mathcal{H}) \sqcup \Omega(\mathcal{H}))$ be the canonical isomorphism. Consider the following diagram:

$$\begin{array}{ccccc}
 \Omega(\mathcal{H}) & \xrightarrow{\varepsilon} & M(\Omega(\mathcal{H})) & \xrightarrow{q^{-1} \circ M(\delta)} & M(\Omega(\mathcal{H})) \otimes M(\Omega(\mathcal{H})) \\
 \searrow i & & \downarrow \bar{i} & [2] & \downarrow \bar{i} \otimes \bar{i} \\
 & & H & \xrightarrow{\Delta} & H \otimes H
 \end{array}$$

I would like diagram [2] to commute. Diagram [1] commutes due to the hypothesis. Let diagram [1]+[2] be the large diagram consisting of the outside arrows.

Claim: Diagram [1]+[2] commutes.

Proof: $\varepsilon(\Omega(\mathcal{H})) \subset \Omega(M(\Omega(\mathcal{H})))$ by 25.8. So if $h \in \Omega(\mathcal{H})$, then $[q^{-1} \circ M(\delta)][\varepsilon(h)] = \varepsilon(h) \otimes \varepsilon(h)$. So recalling that diagram [1] commutes, we find that

$$[(\bar{i} \otimes \bar{i}) \circ (q^{-1} \circ M(\delta)) \circ \varepsilon](h) = \bar{i}(\varepsilon(h)) \otimes \bar{i}(\varepsilon(h)) = i(h) \otimes i(h).$$

But also $\Delta(i(h)) = i(h) \otimes i(h)$ since $h \in \Omega(\mathcal{A})$. Thus diagram [1]+[2] commutes. QED on claim

But $i = \bar{i} \circ \varepsilon$, so $(\Delta \circ \bar{i}) \circ \varepsilon = [(\bar{i} \otimes \bar{i}) \circ q^{-1} \circ M(\delta)] \circ \varepsilon$. But by the uniqueness statement of 10.6, this implies that $\Delta \circ \bar{i} = (\bar{i} \otimes \bar{i}) \circ (q^{-1} \circ M(\delta))$, i.e. diagram [2] commutes. Hence \bar{i} is comultiplicative.

Consider:

$$\begin{array}{ccccc}
 & & & & H \\
 & & & & \downarrow 1 \\
 & & & & H \\
 \Omega(\mathcal{A}) & \xrightarrow{\varepsilon} & M(\Omega(\mathcal{A})) & \xrightarrow{\bar{i}} & H \\
 \downarrow \lambda & & \downarrow M(\lambda) & & \downarrow \zeta \\
 \{1\} & \xrightarrow{\varepsilon} & M(\{1\}) & \xrightarrow{j^{-1}} & \mathbb{K}
 \end{array}$$

[1] [2] [3]

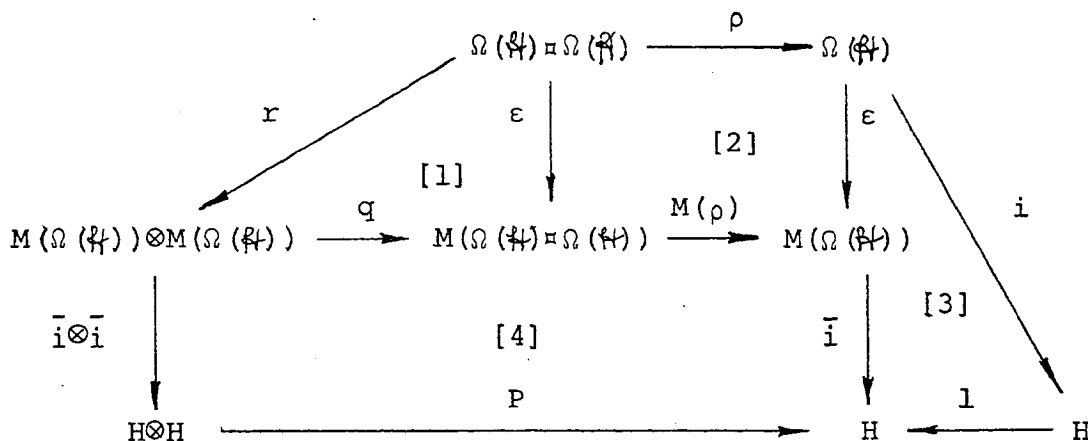
I would like diagram [3] to commute. We know that diagram [2] commutes because of the definition of $M(\lambda)$. Diagram [1] commutes by hypothesis. Let [1]+[2]+[3] be the large diagram formed by the outside arrows. I claim that it commutes. By 25.3a, $(\zeta \circ i)(h) = 1$ for all $h \in \Omega(\mathcal{A})$. Also $(j^{-1} \circ \varepsilon \circ \lambda)(h) = j^{-1}(\varepsilon(\lambda(h))) = j^{-1}(\varepsilon(1)) = 1$ for all $h \in \Omega(\mathcal{A})$. So $\zeta \circ i = j^{-1} \circ \varepsilon \circ \lambda$. Hence [1]+[2]+[3] commutes.

These facts imply that $j^{-1} \circ M(\lambda) \circ \varepsilon = \zeta \circ \bar{i} \circ \varepsilon$. Thus again by the uniqueness of maps (cf. 10.6), $j^{-1} \circ M(\lambda) = \zeta \circ \bar{i}$. Hence diagram [3] commutes. Hence \bar{i} preserves counits.

Hence \bar{i} is a \mathcal{D} -coalgebra morphism.

Now assume $\mathcal{H} = (H, P, U, \Delta, \zeta)$ is a \mathcal{D} -bialgebra. Let $\rho: \Omega(\mathcal{H}) \square \Omega(\mathcal{H}) \rightarrow \Omega(\mathcal{H})$ be the multiplication map of $\Omega(\mathcal{H})$, i.e. $\rho(h, h') = P(h \otimes h')$. Let $u: \{1\} \rightarrow \Omega(\mathcal{H})$ be defined by $u(1) = U(1)$ (recall that $U(1)$ is the identity of the monoid $\Omega(\mathcal{H})$).

Consider:



where r is as in 23.11. Diagram [2] commutes because of the definition of $M(\rho)$, [3] commutes by hypothesis, and [1] commutes by 23.11. Let [1]+[2]+[3]+[4] be the large diagram which is formed by the outside arrows. I claim that it commutes. Let $(h, h') \in \Omega(\mathcal{H}) \square \Omega(\mathcal{H})$. Thus recalling that [3] commutes, we have that

$$\begin{aligned}
 [P \circ (\bar{i} \otimes \bar{i}) \circ r](h, h') &= P([\bar{i} \circ \varepsilon](h) \otimes [\bar{i} \circ \varepsilon](h')) = P(i(h) \otimes i(h')) \\
 &= P(h \otimes h'). \text{ On the other hand,} \\
 i(\rho(h, h')) &= i(P(h \otimes h')) = P(h \otimes h'). \text{ So [1]+[2]+[3]+[4]} \\
 &\text{commutes.}
 \end{aligned}$$

All of these facts imply that

$$\begin{aligned}
 [P \circ (\bar{i} \otimes \bar{i})] \circ r &= [\bar{i} \circ M(\rho) \circ q] \circ r. \text{ Hence by 18.1,} \\
 P \circ (\bar{i} \otimes \bar{i}) &= \bar{i} \circ M(\rho) \circ q. \text{ Hence diagram [4] commutes. Hence} \\
 \bar{i}: M(\Omega(\mathcal{H})) &\rightarrow \mathcal{H} \text{ is multiplicative.}
 \end{aligned}$$

Consider:

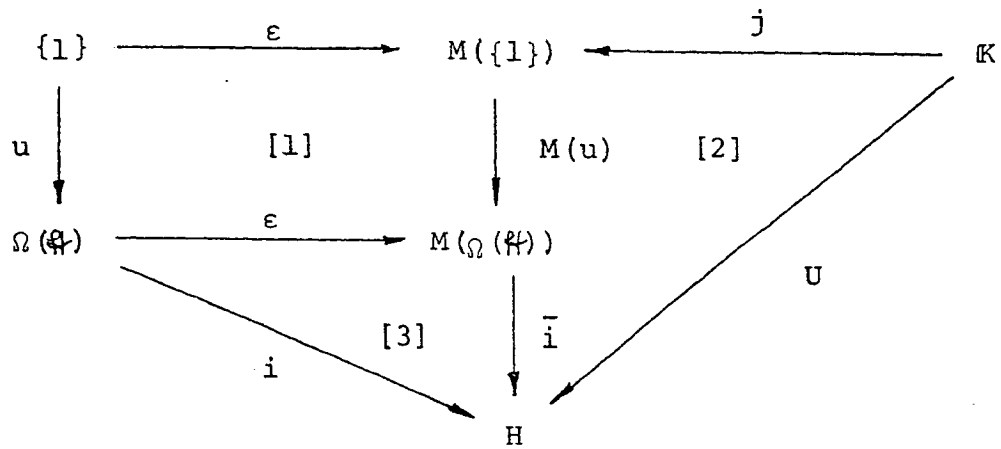


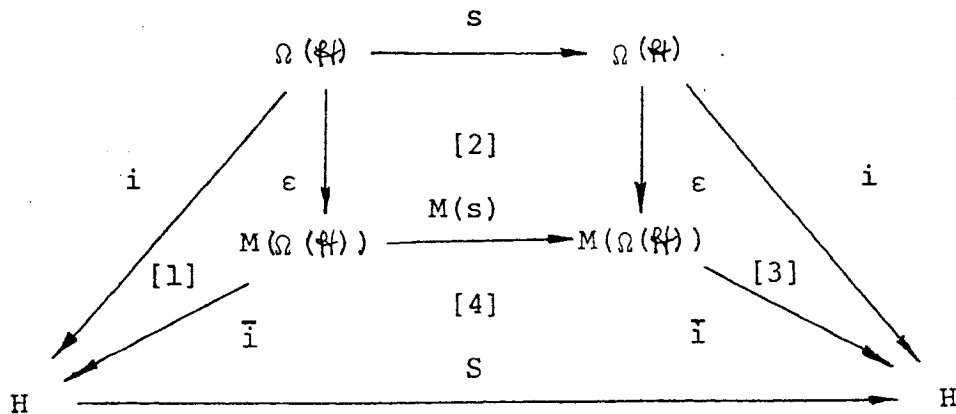
Diagram [3] commutes by hypothesis and diagram [1] commutes by the definition of $M(u)$. Diagram [2] commutes since

$$\bar{i}(M(u)[j(1)]) = \bar{i}(M(u)[\epsilon(1)]) = \bar{i}(\epsilon[u(1)]) = i[U(1)] = U(1).$$

Thus since 1 spans \mathbb{K} , $U = \bar{i} \circ (M(u) \circ j)$. Hence \bar{i} preserves units. Hence \bar{i} is a \mathcal{D} -algebra morphism. Hence \bar{i} is a \mathcal{D} -bialgebra morphism.

Lastly assume $\mathcal{H} = (H, P, U, \Delta, \zeta, S)$ is a \mathcal{D} -Hopf algebra. Define $s: \Omega(\mathcal{H}) \rightarrow \Omega(\mathcal{H})$ by $s(h) = S(h)$. Recall from the proof of 25.3 that s is the inverse map for $\Omega(\mathcal{H})$.

Consider:



Diagrams [1] and [3] commute by hypothesis, and diagram [2] commutes because of the definition of $M(s)$. The large diagram (formed by the outside arrows) commutes just by the definition of i and of s . All of these facts implies that $(S \circ \bar{i}) \circ \varepsilon = (\bar{i} \circ M(s)) \circ \varepsilon$. So again by uniqueness (cf. 10.6), $S \circ \bar{i} = \bar{i} \circ M(s)$, i.e. diagram [4] commutes. Hence \bar{i} preserves antipodes and hence \bar{i} is a \mathcal{D} -Hopf algebra morphism QED on 25.10a

Now to continue with the proof of 25.10.

Suppose G is a k -group (k -monoid or k -space respectively). By 24.24 and 25.8, we see that $M(G)$ is an object in \mathcal{L} (\mathcal{M} or \mathcal{N} respectively) and that $\varepsilon: G \rightarrow \Omega(M(G))$ is a morphism in \mathcal{R} (\mathcal{S} or \mathcal{J} respectively). So that all I must do is show that if \mathcal{H} is an object in \mathcal{L} (\mathcal{M} or \mathcal{N} respectively) and if $f: G \rightarrow \Omega(\mathcal{H})$ is a morphism in \mathcal{R} (\mathcal{S} or \mathcal{J} respectively), then there exists a unique \mathcal{L} (\mathcal{M} or \mathcal{N} respectively) morphism $\bar{f}: M(G) \rightarrow \mathcal{H}$ such that

$$\begin{array}{ccc}
 G & \xrightarrow{f} & \Omega(\mathcal{H}) \\
 \varepsilon \downarrow & \nearrow & \\
 \Omega(M(G)) & & \Omega(\bar{f})
 \end{array}$$

commutes. So suppose \mathcal{H} is a non-zero \mathcal{D} -Hopf algebra (non-zero \mathcal{D} -bialgebra, or \mathcal{D} -coalgebra respectively) with underlying p -reflexive space H and $f: G \rightarrow \Omega(\mathcal{H})$ is a k -group (k -monoid or k -space respectively) morphism. Let $i: \Omega(\mathcal{H}) \rightarrow H$ be the

inclusion map. Let $\bar{i}: M(\Omega(\mathcal{H})) \rightarrow H$ be the unique continuous linear map so that $\bar{i} \circ \varepsilon = i$. Consider

$$\begin{array}{ccccc}
 G & \xrightarrow{f} & \Omega(\mathcal{H}) & & \\
 \varepsilon \downarrow & & \varepsilon \downarrow & \searrow i & \\
 & [1] & & & \\
 M(G) & \xrightarrow{M(f)} & M(\Omega(\mathcal{H})) & \xrightarrow{\bar{i}} & H \\
 & & [2] & &
 \end{array}$$

Diagram [1] commutes because of the definition of $M(f)$. Diagram [2] commutes because of the comments above. So the large diagram [1]+[2], formed by the outside arrows, commutes. $M(f)$ is a \mathcal{D} -Hopf algebra (\mathcal{D} -bialgebra or \mathcal{D} -coalgebra respectively) morphism, since M is a functor (cf. 24.24). But \bar{i} is as well, by 25.10a. So $\bar{i} \circ M(f)$ is a morphism in $\mathcal{L}(\mathcal{M}$ or \mathcal{N} respectively). Define $\bar{f} = \bar{i} \circ M(f)$. Then if $x \in G$, $\varepsilon(x) \in \Omega(M(G))$ by 25.8. So $[\Omega(\bar{f})](\varepsilon(x)) = \bar{f}(\varepsilon(x)) = i(f(x)) = f(x)$, because $\Omega(\bar{f}) = \bar{f}|_{\Omega(M(G))}$. So $\Omega(\bar{f}) \circ \varepsilon = f$.

Suppose $g: M(G) \rightarrow \mathcal{H}$ is another $\mathcal{L}(\mathcal{M}$ or \mathcal{N} respectively) morphism such that $\Omega(g) \circ \varepsilon = f$. Then $g \circ \varepsilon = i \circ f$, since $\Omega(g) = g|_{\Omega(M(G))}$ and since $\varepsilon(G) \subset \Omega(M(G))$. But also $\bar{f} \circ \varepsilon = i \circ f$ since diagram [1]+[2] above commutes. So since there is at most one continuous linear map such that the diagram

$$\begin{array}{ccc}
 G & \xrightarrow{i \circ f} & H \\
 \varepsilon \downarrow & \nearrow & \\
 M(G) & &
 \end{array}$$

commute, we have $\bar{f} = g$. So we have proved that there is exactly one morphism which makes the following diagram commute:

$$\begin{array}{ccc}
 G & \xrightarrow{f} & \Omega(\mathcal{H}) \\
 \varepsilon \downarrow & \nearrow \Omega(-) & \\
 \Omega(M(G)) & &
 \end{array}$$

QED on 25.10

Section 26 - A natural conjugation on M .

In this section it is shown that a "natural conjugation" can be associated with the functor M ; i.e. for all k -spaces X , there is a unique continuous conjugate linear function ϕ from $M(X)$ to $M(X)$ which when restricted to point evaluations acts as the identity map.

This concept while interesting in itself, will be used later to define an involution on $M(G)$ where G is a k -group.

26.0 Remark. If $\mathbb{K} = \mathbb{R}$, then everything in this section is trivial (cf. 22.2). However, it will be of use to carry the real case along.

26.1 Theorem. Let $H: \mathcal{J} \rightarrow \mathcal{J}$ be the "conjugate space functor" as described in 22.1. Then there exists a natural isomorphism $\phi: M \rightarrow H \circ M$ such that for all $X \in \mathcal{J}$, $H(\phi(X)) \circ \phi(X) = 1_{M(X)}$ and $\phi(X) \circ H(\phi(X)) = 1_{[H \circ M](X)}$.

26.2 Comment. Theorem 26.1 is perhaps best expressed by saying that the functor M has a natural conjugation associated with it.

Proof of 26.1: Let X be a k -space. Let $\epsilon: X \rightarrow M(X)$ be the canonical map. Recall that as topological spaces $M(X)$ and $H(M(X))$ are the same.

So define a map $f: X \rightarrow H(M(X))$ by $f(x) = \varepsilon(x)$. f is a continuous map, and hence there exists a unique continuous linear map $\phi: M(X) \rightarrow H(M(X))$ such that $\phi \circ \varepsilon = f$.

Consider

$$\begin{array}{ccc}
 X & \xrightarrow{f} & H(M(X)) \\
 \varepsilon \downarrow & \nearrow \phi & \downarrow H(\phi) \\
 M(X) & \xrightarrow{1_{M(X)}} & M(X) = H(H(M(X)))
 \end{array}
 \begin{array}{l}
 [1] \\
 [2]
 \end{array}$$

Diagram [1] commutes by the definition of ϕ . The outside diagram [1]+[2] (formed by the outside arrows) commutes because $[H(\phi) \circ f](t) = \phi(\varepsilon(t)) = f(t) = \varepsilon(t)$ for all $t \in X$. So $H(\phi) \circ f = 1_{M(X)} \circ \varepsilon$. Hence

$$\begin{aligned}
 [H(\phi) \circ \phi] \circ \varepsilon &= 1_{M(X)} \circ \varepsilon. \text{ Thus by uniqueness we have} \\
 H(\phi) \circ \phi &= 1_{M(X)}. \text{ Applying } H \text{ we find that} \\
 \phi \circ H(\phi) &= H(H(\phi)) \circ H(\phi) = H(H(\phi) \circ \phi) = H(1_{M(X)}) = 1_{H(M(X))}.
 \end{aligned}$$

Hence we have proved that for all $X \in \mathcal{J}$, there exists an isomorphism $\phi: M(X) \rightarrow H(M(X))$ such that $H(\phi) \circ \phi = 1_{M(X)}$ and $\phi \circ H(\phi) = 1_{H(M(X))}$.

We only need prove that the collection of ϕ 's are natural.

Suppose X and Y are k -spaces and $g: X \rightarrow Y$ is a continuous map. Consider the following diagram:

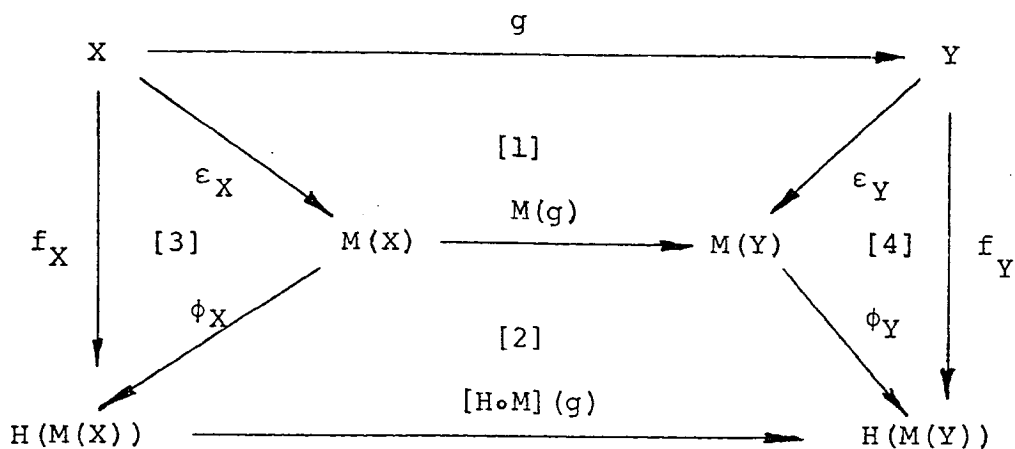


Diagram [1] commutes because of the definition of $M(g)$.
 Diagram [3] and [4] commute because of the definition of ϕ .
 Let us now consider the large diagram formed by the outside arrows. Let $t \in X$. Then

$$H(M(g))(f_X(t)) = M(g)(\epsilon_X(t)) = \epsilon_Y(g(t)) = f_Y(g(t)).$$

So this outside diagram commutes. Hence

$$[\phi_Y \circ M(g)] \circ \epsilon_X = (H(M(g)) \circ \phi_X) \circ \epsilon_X.$$

Again by uniqueness, we have $\phi_Y \circ M(g) = H(M(g)) \circ \phi_X$. Hence diagram [2] commutes.

Thus the ϕ 's are natural.

QED

The following proposition will be of value in computations.

26.3 Proposition. Let X be a k -space, then

$\phi_X: M(X) \rightarrow H(M(X))$ is defined by

$$[\phi_X(\mu)](g) = \sigma(\mu(\sigma \circ g)) \quad \text{for all } \mu \in M(X) \text{ and } g \in c(X),$$

where $\sigma: \mathbb{K} \rightarrow \mathbb{K}$ represents the complex conjugation function.

Proof: Let $\ell: H(C(X)) \rightarrow c_c(X)$ be defined by $\ell(f) = \sigma \circ f$. Note that ℓ is linear. In order to show that ℓ is continuous, let $f_\alpha \rightarrow 0$ in $H(C(X))$, i.e.

in $C(X)$. Then $f_\alpha \rightarrow 0$ in $c_c(X)$. So $\sigma \circ f_\alpha = \ell(f_\alpha)$ converges to 0 on compact sets, since σ is continuous. Hence $\ell: H(C(X)) \rightarrow c_c(X)$ is continuous and linear. $H(C(X))$ is p -determined by 22.4. So by the coreflection property, ℓ is continuous from $H(C(X))$ to $C(X)$. Also note that σ is a continuous linear map from \mathbb{K} to $H(\mathbb{K})$.

Recalling that Hom is a functor from $\mathcal{D}^{\text{op}} \times \mathcal{D}$ to \mathcal{D} , $\text{Hom}(\ell, \sigma)$ is a continuous linear map from $\text{Hom}(C(X), \mathbb{K})$ to $\text{Hom}(H(C(X)), H(\mathbb{K}))$ defined by $\mu \mapsto \sigma \circ \mu \circ \ell$. Recalling that $M(X) = \text{Hom}(C(X), \mathbb{K})$ and $\text{Hom}(H(C(X)), H(\mathbb{K})) = H(\text{Hom}(C(X), \mathbb{K}))$ (cf. 22.7), we see that $\text{Hom}(\ell, \sigma)$ is a continuous linear map from $M(X)$ to $H(M(X))$ such that for $g \in c(X)$

$$[[\text{Hom}(\ell, \sigma)](\mu)](g) = \sigma(\mu(\ell(g))) = \sigma(\mu(\sigma \circ g)).$$

Now let ε_X , f_X , and ϕ_X be as in the proof of 26.1. Let $g \in c(X)$. Then if $t \in X$

$$[[\text{Hom}(\ell, \sigma)](\varepsilon_X(t))](g) = \sigma([\varepsilon_X(t)](\sigma \circ g)) = \sigma(\sigma(g(t))) = g(t) = [f_X(t)](g).$$

So $\text{Hom}(\ell, \sigma) \circ \varepsilon_X = f_X$. But by definition, ϕ_X is the unique continuous linear map with this property. Hence $\phi_X = \text{Hom}(\ell, \sigma)$. QED

The following two propositions relate the level of k -spaces with the level of p -reflexive spaces. The first proposition says essentially that $\phi: M \rightarrow H \circ M$ commutes with "tensor" products; and the second says that when $M(\{1\})$ is identified with the scalars, then ϕ is just conjugation of complex numbers.

26.4 Proposition. Let $\phi: M \rightarrow H \circ M$ be as described in 26.1. Let $q: \otimes_0(M \times M) \rightarrow M \circ \square$ be the natural isomorphism described in 23.10. Let $\psi: \otimes_0(H \times H) \rightarrow H \circ \otimes$ be the natural isomorphism described in 22.8. Then for all k -spaces X and Y , the following diagram commutes:

$$\begin{array}{ccc}
 M(X) \otimes M(Y) & \xrightarrow{\phi \otimes \phi} & H \circ M(X) \otimes H \circ M(Y) \\
 \downarrow q & & \downarrow \psi \\
 & & H(M(X) \otimes M(Y)) \\
 & & \downarrow H(q) \\
 M(X \square Y) & \xrightarrow{\phi} & H \circ M(X \square Y)
 \end{array}$$

Proof: Since the linear spans of the point evaluations of X and Y are dense in $M(X)$ and $M(Y)$ respectively, by 18.1 $\{\varepsilon(x) \otimes \varepsilon(y) : x \in X \text{ and } y \in Y\}$ has a dense linear span in $M(X) \otimes M(Y)$. Hence it suffices to check the commutativity of the diagram on elements of this set whose span is dense. By 23.11, $q(\varepsilon(x) \otimes \varepsilon(y)) = \varepsilon(x, y)$ for all $(x, y) \in X \square Y$.

We thus see from the definitions of ϕ , ψ , and H , that the diagram commutes. QED

26.5 Proposition. Let $\sigma: \mathbb{K} \rightarrow H(\mathbb{K})$ be the complex conjugation function. Then the following diagram commutes:

$$\begin{array}{ccc}
 \mathbb{K} & \xrightarrow{\sigma} & H(\mathbb{K}) \\
 \downarrow j & & \downarrow H(j) \\
 M(\{1\}) & \xrightarrow{\phi} & H(M(\{1\}))
 \end{array}$$

Proof: Since $\{1\}$ is a basis for \mathbb{K} , it is sufficient to check that $(H(j) \circ \sigma)(1) = (\phi \circ j)(1)$. But this is true since $\sigma(1) = 1$ and $\phi(\varepsilon(1)) = [H(j)](1) = j(1) = \varepsilon(1)$. QED

Section 27 - Representation theory.

In this final section \mathcal{D} -algebras-with-involution are defined and it is shown that M is a functor from k -groups to \mathcal{D} -algebras-with-involution. It is also shown that if G is a k -monoid, then the category of G -representations is isomorphic to the category of $M(G)$ -representations; and if G is a k -group, then the category of unitary G -representations is isomorphic to the category of "star" $M(G)$ -representations.

27.1 Definition. A four-tuple (A, P, U, i) will be called a \mathcal{D} -algebra-with-involution provided (A, P, U) is a \mathcal{D} -algebra and $i: A \rightarrow H(A)$ is a continuous linear map such that the following three diagrams commute:

$$\begin{array}{ccccc}
 A \otimes A & \xrightarrow{c} & A \otimes A & \xrightarrow{i \otimes i} & H(A) \otimes H(A) \\
 \downarrow P & & & & \downarrow \psi \\
 & & & & H(A \otimes A) \\
 & & & & \downarrow H(P) \\
 A & \xrightarrow{i} & & & H(A)
 \end{array}$$

where $c: A \otimes A \rightarrow A \otimes A$ is the natural commutativity isomorphism of 18.3 and $\psi: H(A) \otimes H(A) \rightarrow H(A \otimes A)$ is the natural isomorphism described in 22.8.

$$\begin{array}{ccc}
 \mathbb{K} & \xrightarrow{\sigma} & H(\mathbb{K}) \\
 \downarrow U & & \downarrow H(U) \\
 A & \xrightarrow{i} & H(A)
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{1} & A \\
 \searrow i & & \nearrow H(i) \\
 & H(A) &
 \end{array}$$

where $\sigma: \mathbb{K} \rightarrow H(\mathbb{K})$ is the complex conjugation function.

Thus the involution map i is an anti-algebra morphism of order ≤ 2 which preserves the unit. The fact that i is linear from A to $H(A)$ implies that i is conjugate linear.

27.2 Definition. Let $\mathcal{A} = (A, P, U, i)$ and $\mathcal{A}' = (A', P', U', i')$ be \mathcal{D} -algebras-with-involution. Then a continuous linear map $f: A \rightarrow A'$ will be called a \mathcal{D} -algebra-with-involution morphism from \mathcal{A} to \mathcal{A}' provided f is a \mathcal{D} -algebra morphism from (A, P, U) to (A', P', U') and provided the following diagram commutes:

$$\begin{array}{ccc}
 A & \xrightarrow{f} & A' \\
 \downarrow i & & \downarrow i' \\
 H(A) & \xrightarrow{H(f)} & H(A')
 \end{array}$$

27.3 Definition. A \mathcal{D} -algebra or a \mathcal{D} -algebra-with-involution will be called non-zero provided its underlying topological vector space is non-zero.

27.4 Definition. Let \mathcal{O} = the category of non-zero \mathcal{D} -algebras-with-involution and \mathcal{D} -algebra-with-involution morphisms; and let \mathcal{P} = the category of non-zero

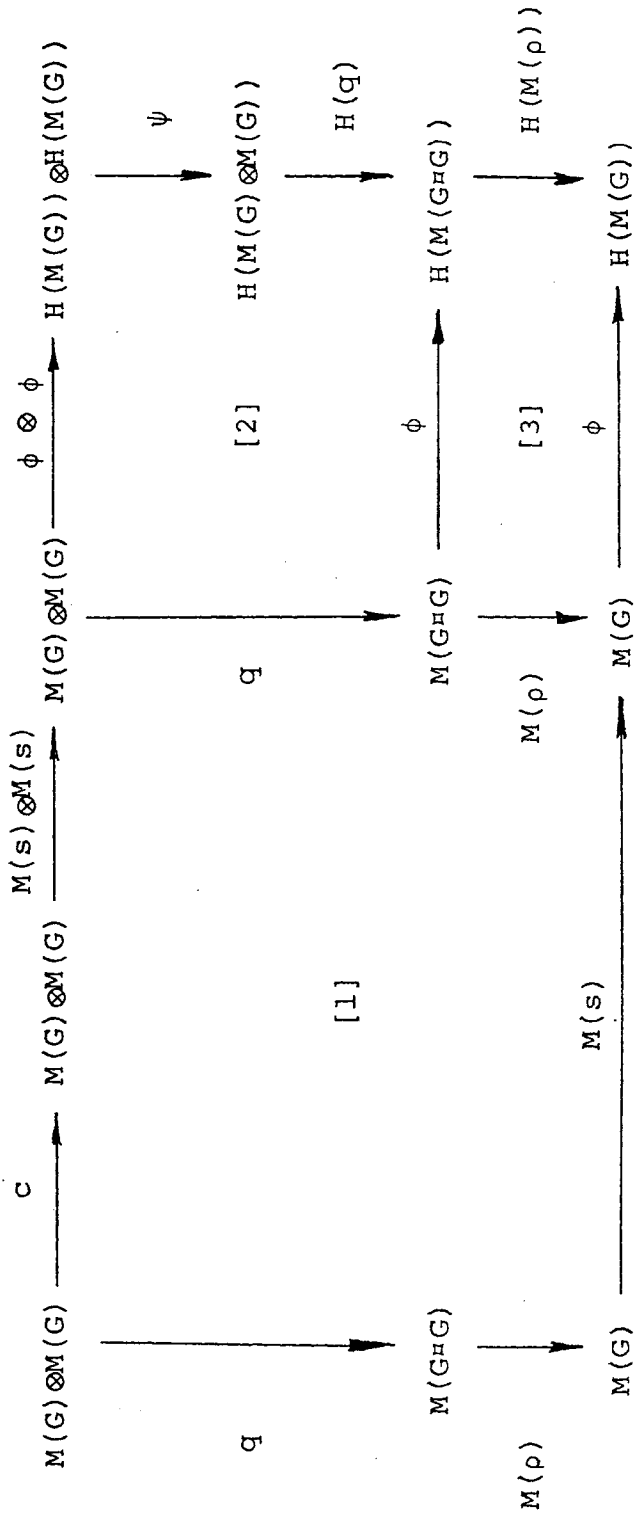
\mathcal{D} -algebras and \mathcal{D} -algebra morphisms.

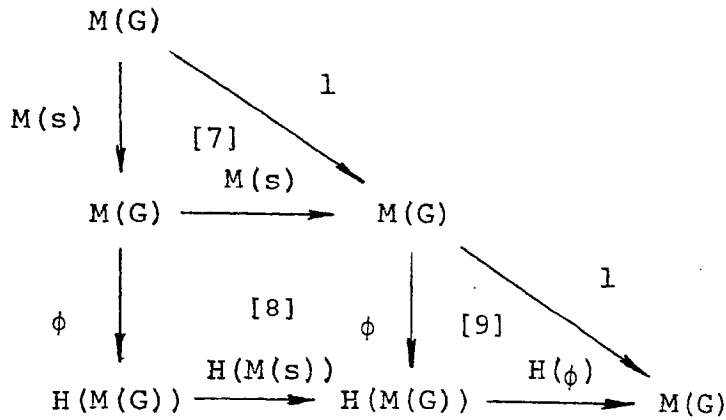
27.5 Theorem. If G is a k -group (respectively, k -monoid), then the structure of G induces on $M(G)$ the structure of a non-zero \mathcal{D} -algebra-with-involution (respectively, non-zero \mathcal{D} -algebra) such that $M: \mathcal{A} \rightarrow \mathcal{O}$ (respectively, $M: \mathcal{J} \rightarrow \mathcal{P}$) is a functor.

Proof: Assume G is a k -group. Let $\delta: G \rightarrow G \times G$, $\lambda: G \rightarrow \{1\}$, $\rho: G \times G \rightarrow G$, $u: \{1\} \rightarrow G$, and $s: G \rightarrow G$ be as in the proof of 24.24. Then by that proof, we find that $(M(G), M(\rho) \circ q, M(u) \circ j, q^{-1} \circ M(\delta), j^{-1} \circ M(\lambda), M(s))$ is a non-zero \mathcal{D} -Hopf algebra. In particular, $(M(G), M(\rho) \circ q, M(u) \circ j)$ is a non-zero \mathcal{D} -algebra.

Now consider the following three large diagrams (each consisting of various "subdiagrams"):

$$\begin{array}{ccccc}
 M(G) & \xrightarrow{M(s)} & M(G) & \xrightarrow{\phi} & H(M(G)) \\
 & \swarrow M(u) & & \nearrow M(u) & \uparrow H(M(u)) \\
 & & [4] & & [5] \\
 & & M(\{1\}) & \xrightarrow{\phi} & H(M(\{1\})) \\
 & \uparrow j & & & \uparrow H(j) \\
 & & [6] & & \\
 K & \xrightarrow{\sigma} & & & H(K)
 \end{array}$$





Diagrams [1], [4], and [7] commute essentially because $M(G)$ is a cocommutative Hopf algebra (cf. 24.24). Diagrams [3], [5], and [8] commute because ϕ is natural. Diagrams [2], [6], and [9] commute due to 26.4, 26.5, and 26.1 respectively. Hence the three large diagrams, consisting of the outside arrows in each case, commute.

Hence $(M(G), M(\rho) \circ q, M(u) \circ j, \phi \circ M(s))$ is a \mathcal{D} -algebra-with-involution. Thus M behaves properly with respect to objects. Let us check the situation with respect to morphisms.

Suppose G and G' are k -groups and $f: G \rightarrow G'$ is a k -group morphism. Then by 24.24, $M(f)$ is a \mathcal{D} -Hopf algebra morphism from $M(G)$ to $M(G')$. In particular, $M(f)$ is an algebra morphism from $M(G)$ to $M(G')$; and diagram [1] of the following diagram commutes, where $s: G \rightarrow G$ and $s': G' \rightarrow G'$ are each defined by $x \mapsto x^{-1}$:

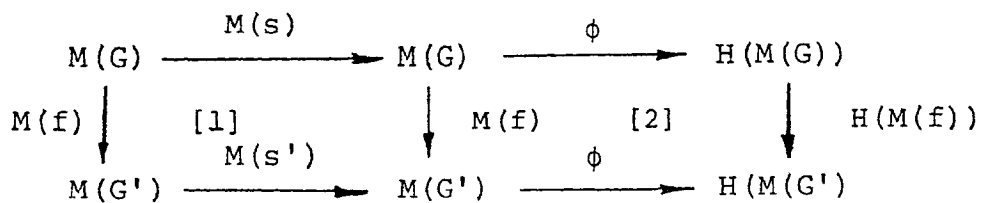


Diagram [2] commutes since ϕ is natural. Thus diagram [1]+[2], i.e. the one formed by the outside arrows, commutes. Hence $M(f)$ is a \mathcal{D} -algebra-with-involution morphism.

Since $M: \mathcal{J} \rightarrow \mathcal{D}$ is a functor we know that it behaves properly with respect to composition and identities. Thus $M: \mathcal{K} \rightarrow \mathcal{G}$ is a functor.

That $M: \mathcal{J} \rightarrow \mathcal{P}$ is a functor follows trivially from the fact that $M: \mathcal{J} \rightarrow \mathcal{M}$ is a functor (cf. 24.24).

QED

The following proposition will be useful for computations.

27.6 Proposition. Let G be a k -group. Let $\mu \mapsto \mu^*$ denote the involution map on $M(G)$ (cf. 27.5). Then

- 1) if $\mu \in M(G)$,
 $[\mu^*](f) = \sigma(\mu(x \mapsto \sigma(f(x^{-1}))))$ for all $f \in c(G)$, where $\sigma: \mathbb{K} \rightarrow \mathbb{K}$ is the complex conjugation function; and
- 2) if $\varepsilon: G \rightarrow M(G)$ is the canonical map, then
 $[\varepsilon(x)]^* = \varepsilon(x^{-1})$ for all $x \in G$.

Proof: 1) By definition, $(-)^* = \phi \circ M(s)$ where $s: G \rightarrow G$ is defined by $s(g) = g^{-1}$ and $\phi: M \rightarrow H \circ M$ is the natural transformation described in 26.1.

Let $\mu \in M(G)$ and $f \in c(G)$. Then
 $[[\phi \circ M(s)](\mu)](f) = [\phi(M(s)(\mu))](f) = \sigma([M(s)(\mu)](\sigma \circ f))$
 by 26.3. But $[M(s)(\mu)](\sigma \circ f) = \mu(\sigma \circ f \circ s)$. So
 $[\mu^*](f) = \sigma(\mu(\sigma \circ f \circ s)) = \sigma(\mu(x \mapsto \sigma(f(x^{-1}))))$.

2) Again note that $(-)^* = \phi \circ M(s)$. By definition of $M(s)$, $M(s) \circ \varepsilon = \varepsilon \circ s$. So

$$[M(s)](\varepsilon(x)) = \varepsilon(s(x)) = \varepsilon(x^{-1}) \quad \text{for all } x \in G. \quad \text{Also}$$

recall that $\phi: M(G) \rightarrow H(M(G))$ has the property that $\phi(\varepsilon(t)) = \varepsilon(t)$ for all $t \in G$ (cf. 26.1). So

$$[\varepsilon(x)]^* = \phi([M(s)](\varepsilon(x))) = \phi(\varepsilon(x^{-1})) = \varepsilon(x^{-1}) \quad \text{for all}$$

$x \in G$.

QED

Following are a series of technical propositions which will be needed in our discussion of representation theory. They essentially say that the representations which I define will in nice cases agree with the standard strongly continuous representations.

27.7 Proposition. Let X be a k -space and V be a vector space. Let \mathcal{U}_1 and \mathcal{U}_2 be two locally convex Hausdorff topologies on V such that (a) $\mathcal{U}_1 \subset \mathcal{U}_2$ and (b) (V, \mathcal{U}_2) is p -complete. Suppose $L: M(X) \rightarrow V$ is a linear map such that (1) $L: M(X) \rightarrow (V, \mathcal{U}_1)$ is continuous and (2) for all compact subsets K of $M(X)$, $L|_K: K \rightarrow (V, \mathcal{U}_2)$ is continuous. Then $L: M(X) \rightarrow (V, \mathcal{U}_2)$ is continuous.

Proof: Let $\varepsilon: X \rightarrow M(X)$ be the canonical map.

If Q is a compact subset of X , then $\varepsilon(Q)$ is compact in $M(X)$, since ε is continuous. Thus

$$(L \circ \varepsilon)|_Q: Q \rightarrow (V, \mathcal{U}_2) \quad \text{is continuous by property (2).}$$

But X is a k -space, so $L \circ \varepsilon: X \rightarrow (V, \mathcal{U}_2)$ is continuous. Since (V, \mathcal{U}_2) is p -complete, there exists a unique continuous linear map $L': M(X) \rightarrow (V, \mathcal{U}_2)$ such

that $L' \circ \varepsilon = L \circ \varepsilon$. Now since $\mathcal{V}_1^p \subset \mathcal{V}_2^p$, $L': M(X) \rightarrow (V, \mathcal{V}_1^p)$ is continuous. So $L - L'$ is continuous from $M(X)$ to (V, \mathcal{V}_1^p) . Hence $(L - L')^{-1}(\{0\})$ is a closed linear subspace of $M(X)$, since (V, \mathcal{V}_1^p) is Hausdorff, which contains $\{\varepsilon(x) : x \in X\}$, since $L' \circ \varepsilon = L \circ \varepsilon$. But the closed linear span of $\{\varepsilon(x) : x \in X\}$ equals $M(X)$. Hence $M(X) \subset (L - L')^{-1}(\{0\})$. So $L = L'$. Hence L is continuous from $M(X)$ to (V, \mathcal{V}_2^p) . QED

27.8 Proposition. Let X be a k -space. Let V and W be locally convex spaces such that V is barrelled and W is p -complete. If $L: M(X) \rightarrow \text{hom}(V, W)$ is a linear map which is continuous when $\text{hom}(V, W)$ is given the topology of pointwise convergence, then $L: M(X) \rightarrow \text{Hom}(V, W)$ is continuous.

Proof: Let K be a compact subset of $M(X)$. $L(K)$ is then compact hence bounded in $\text{hom}(V, W)$ with the topology of pointwise convergence. By theorem 2 of §3, chap. 3 of [4], $L(K)$ is equicontinuous. By prop. 5 of §3, chap. 3 of [4], the relative uniformities and hence the relative topologies induced on $L(K)$ by the topology of simple convergence and the topology of precompact convergence agree. Thus $L|_K: K \rightarrow \text{hom}_p(V, W)$ is continuous.

Now note that $\text{hom}_p(V, W)$ is p -complete by 1.23 and that the topology of pointwise convergence is coarser than the topology of precompact convergence. Hence by 27.7,

$L: M(X) \rightarrow \text{hom}_p(V, W)$ is continuous. But $M(X)$ is p -determined, so $L: M(X) \rightarrow \alpha(\text{hom}_p(V, W)) = \text{Hom}(V, W)$ is continuous. QED

27.9 Corollary to proof. Let X be a k -space, V be a barrelled locally convex space, and W be a p -complete locally convex space. If $f: X \rightarrow \text{hom}(V, W)$ is continuous when $\text{hom}(V, W)$ is given the topology of pointwise convergence, then $f: X \rightarrow \text{Hom}(V, W)$ is continuous.

Proof: We see by using essentially the same proof as 27.8 that if $f: X \rightarrow \text{hom}(V, W)$ with pointwise convergence is continuous, then $f: X \rightarrow \text{hom}_p(V, W)$ is continuous. So there exists a unique continuous linear map $F: M(X) \rightarrow \text{hom}_p(V, W)$ such that $F \circ \varepsilon = f$. But since $M(X)$ is p -determined, F is actually continuous from $M(X)$ to $\text{Hom}(V, W) = \alpha(\text{hom}_p(V, W))$. But $f = F \circ \varepsilon$. So f is continuous from X to $\text{Hom}(V, W)$, since ε is continuous. QED

Representations on p -reflexive spaces

27.10 Convention. For the remainder of this section, we will make the following conventions:

If $\mathcal{A} = (A, P, U)$ is a \mathcal{D} -algebra and $x, y \in A$, then we will denote $P(x \otimes y)$ by xy and $U(1)$ will be denoted by I . Also we will frequently suppress mention of P and U and will speak of the \mathcal{D} -algebra A .

27.11 Definition. If G is a k -monoid, then a

generalized G-representation is a \mathcal{D} -algebra A together with a continuous function $R: G \rightarrow A$ such that $R(xy) = R(x)R(y)$ for all $x, y \in G$, and such that $R(e) = I$ [where e is the identity of G].

27.12 Definition. If G is a k -monoid, then a generalized $M(G)$ -representation is a \mathcal{D} -algebra A together with a \mathcal{D} -algebra morphism S from the \mathcal{D} -algebra $M(G)$ to A .

27.13 Proposition. Let G be a k -monoid and $\varepsilon: G \rightarrow M(G)$ be the canonical map. Then

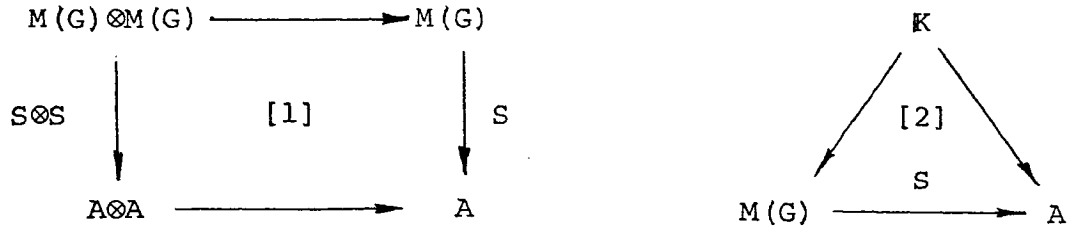
1) $\varepsilon: G \rightarrow M(G)$ is a generalized G -representation; and 2) If $R: G \rightarrow A$ is a generalized G -representation, then there exists a unique generalized $M(G)$ -representation $S: M(G) \rightarrow A$ such that the following diagram commutes:

$$\begin{array}{ccc}
 G & \xrightarrow{R} & A \\
 \varepsilon \downarrow & \nearrow S & \\
 M(G) & &
 \end{array}$$

Proof: We know that ε is continuous; and the facts that $\varepsilon(x)\varepsilon(y) = \varepsilon(xy)$ and $\varepsilon(e) = I$ are proved in 25.8 (recall that as sets $\Omega(M(G)) \subset M(G)$). This proves 1).

As for 2), suppose $R: G \rightarrow A$ is a generalized G -representation. Let $S: M(G) \rightarrow A$ be the unique continuous linear map such that $S \circ \varepsilon = R$.

Consider the following two diagrams:



By 18.1, since the linear span of $\{\epsilon(x) : x \in G\}$ is dense in $M(G)$, it is sufficient to check the commutativity of diagram [1] on elements of the form $\epsilon(x) \otimes \epsilon(y)$.

$$\begin{aligned}
 \text{But } S(\epsilon(x)\epsilon(y)) &= S(\epsilon(xy)) = R(xy) = R(x)R(y) \\
 &= S(\epsilon(x))S(\epsilon(y))
 \end{aligned}$$

So diagram [1] commutes. Diagram [2] commutes because $I = R(e) = S(\epsilon(e))$ and because it suffices to check commutativity on the basis $\{1\}$ of K . Hence S is a \mathcal{D} -algebra morphism such that $S \circ \epsilon = R$. It is unique since every \mathcal{D} -algebra morphism $S': M(G) \rightarrow A$ such that $S' \circ \epsilon = R$ is also a continuous linear map, and thus $S = S'$ by 10.6. QED

In 24.15, it is proved that if V is p -reflexive, then $\text{Hom}(V, V)$ is a \mathcal{D} -algebra with product $T \otimes S \mapsto T \circ S$ and unit $\alpha \mapsto \alpha|_V$.

27.14 Definition. If G is a k -monoid, then a G -representation is a pair (R, V) such that V is a p -reflexive space and R is a generalized G -representation from G to the \mathcal{D} -algebra $\text{Hom}(V, V)$.

27.15 Definition. If G is a k -monoid, then a

M(G)-representation is a pair (R, V) such that V is a p -reflexive space and R is a generalized $M(G)$ -representation from $M(G)$ to the \mathcal{D} -algebra $\text{Hom}(V, V)$.

We will sometimes suppress the underlying space and speak only of the generalized representation.

The following shows that if V is barrelled and p -complete, for example if V is Frechet, then my definitions of representations agree with the standard notions of "strongly continuous" representations.

27.16 Proposition. Suppose G is a monoid with identity e and V is a barrelled, p -complete space. Then

1) (R, V) is a G -representation iff R is a continuous map from G to $\text{hom}(V, V)$, where $\text{hom}(V, V)$ has the topology of pointwise convergence, such that

$R(xy) = R(x) \circ R(y)$ for all $x, y \in G$ and $R(e) = I$; and

2) (R, V) is a $M(G)$ -representation iff R is a continuous linear map from $M(G)$ to $\text{hom}(V, V)$, where $\text{hom}(V, V)$ has the topology of pointwise convergence, such that $R(\mu\nu) = R(\mu) \circ R(\nu)$ for all $\mu, \nu \in M(G)$ and $R(I) = I$.

Proof: The (\Rightarrow) direction of both 1) and 2) are trivial. Also 27.8 and 27.9 demonstrate that the continuity conditions in both cases are the same. Hence 1) follows immediately from the definition of the product in $\text{Hom}(V, V)$.

Now we will check the (\Leftarrow) direction of 2).

Consider the following two diagrams:

$$\begin{array}{ccc}
 M(G) \otimes M(G) & \xrightarrow{\quad} & M(G) \\
 \downarrow R \otimes R & [1] & \downarrow R \\
 \text{Hom}(V, V) \otimes \text{Hom}(V, V) & \xrightarrow{\quad} & \text{Hom}(V, V)
 \end{array}$$

$$\begin{array}{ccc}
 & K & \\
 & \swarrow & \searrow \\
 M(G) & \xrightarrow{R} & \text{Hom}(V, V)
 \end{array}
 \quad [2]$$

By checking diagram [1] on elementary tensors and by using the fact that $R(\mu\nu) = R(\mu) \circ R(\nu)$, we find that diagram [1] commutes. Using $R(I) = I$, we see that the commutativity of [2] follows from seeing what happens on {1}. QED

27.17 Definition. Let G be a k -monoid. Define a category $\text{Rep}(G)$ as follows:

i) the objects of $\text{Rep}(G)$ will be G -representations; and

ii) If (R, V) and (S, W) are G -representations, then a morphism f from (R, V) to (S, W) will be a continuous linear map from V to W such that $f(R(g)(v)) = S(g)[f(v)]$ for all $g \in G$ and $v \in V$.

27.18 Definition. Let G be a k -monoid. Define a category $\text{Rep}(M(G))$ as follows:

i) the objects of $\text{Rep}(M(G))$ will be $M(G)$ -representations; and

ii) If (R, V) and (S, W) are two $M(G)$ -representations, then a morphism f from (R, V) to (S, W) will be a continuous linear map from V to W such that $f(R(\mu)(v)) = S(\mu)[f(v)]$ for all $\mu \in M(G)$ and $v \in V$.

It is easily verified that $\text{Rep}(G)$ and $\text{Rep}(M(G))$ are in fact categories.

27.19 Proposition. Let G be a k -monoid. Then the categories $\text{Rep}(G)$ and $\text{Rep}(M(G))$ are isomorphic.

Proof: Define a functor $A: \text{Rep}(G) \rightarrow \text{Rep}(M(G))$ as follows:

if (R, V) is an object of $\text{Rep}(G)$, then define $A(R, V)$ to be (T, V) where T is the unique generalized $M(G)$ -representation from $M(G)$ to $\text{Hom}(V, V)$ such that $T \circ \varepsilon = R$ (cf. 27.13). Sometimes I will write merely $A(R) = T$.

if (R, V) and (S, W) are objects in $\text{Rep}(G)$ and $f: (R, V) \rightarrow (S, W)$ is a $\text{Rep}(G)$ -morphism, then define $A(f): A(R, V) \rightarrow A(S, W)$ by $A(f) = f$.

Claim: f is a $\text{Rep}(M(G))$ -morphism.

Proof of claim: Let $v \in V$. Since the topologies of $\text{Hom}(V, V)$ and $\text{Hom}(W, W)$ are finer than the topologies of pointwise convergence, the linear maps $\gamma \mapsto \gamma(v)$ from $\text{Hom}(V, V)$ to V and $\beta \mapsto \beta(f(v))$ from $\text{Hom}(W, W)$ to W are continuous. So the maps $\mu \mapsto f[[A(R)(\mu)](v)]$ and $\mu \mapsto [A(S)(\mu)][f(v)]$ from

$M(G)$ to W are continuous linear maps. Thus because $A(R)[\varepsilon(x)] = R(x)$ and $A(S)(\varepsilon(x)) = S(x)$ we see that when precomposed with $\varepsilon: G \rightarrow M(G)$ these maps agree.

Hence by 10.6, these maps agree everywhere ; i.e.

$$f([A(R)(\mu)](v)) = [A(S)(\mu)](f(v)) \text{ for all } \mu \in M(G).$$

Hence $A(f) = f$ is a $\text{Rep}(M(G))$ -morphism. QED on claim

This completes the definition of A .

It is trivial to verify that A is in fact a functor, i.e. that A behaves properly with respect to composition and identities.

Now define a functor $B: \text{Rep}(M(G)) \rightarrow \text{Rep}(G)$ as follows:

i) If (R, V) is an object in $\text{Rep}(M(G))$, then define $B(R, V) = (R \circ \varepsilon, V)$, where $\varepsilon: G \rightarrow M(G)$ is the canonical map. $R \circ \varepsilon$ is a generalized G -representation since R is a generalized $M(G)$ -representation and since $\varepsilon: G \rightarrow M(G)$ is a generalized G -representation (cf. 27.13). I will sometimes write merely $B(R) = R \circ \varepsilon$.

ii) If (R, V) and (S, W) are objects in $\text{Rep}(M(G))$ and $f: (R, V) \rightarrow (S, W)$ is a $\text{Rep}(M(G))$ -morphism, then define $B(f): B(R, V) \rightarrow B(S, W)$ by $B(f) = f$.

Claim: f is a $\text{Rep}(G)$ -morphism.

Proof of claim: We know that for all $\mu \in M(G)$ and $v \in V$, $f(R(\mu)(v)) = S(\mu)(f(v))$. Hence for all $x \in G$ and $v \in V$, $f(R(\varepsilon(x))(v)) = S(\varepsilon(x))(f(v))$. Thus $f([B(R)(x)](v)) = [B(S)(x)](f(v))$ for all $x \in G$ and $v \in V$. Hence $B(f) = f$ is a $\text{Rep}(G)$ -morphism.

QED on claim

This completes the definition of B .

It is trivial to verify that B is in fact a functor, i.e. that B behaves properly with respect to composition and identities.

Claim: $B \circ A = \text{id}_{\text{Rep}(G)}$ and $A \circ B = \text{id}_{\text{Rep}(M(G))}$

Proof of claim: On morphisms the verification is trivial. As for objects, the definitions of A and B together with 27.13 are all that are needed for an easy verification. QED on claim

QED

Unitary representations on Hilbert space.

27.20 Convention. Let G be a k -group. Recall that by 27.5, $M(G)$ is a \mathcal{D} -algebra-with-involution. Let juxtaposition denote the product in $M(G)$, let I denote the identity of $M(G)$, and let μ^* denote the involution of μ for all $\mu \in M(G)$.

27.21 Definition. Let G be a k -group. A G unitary representation will be an ordered pair (R, H) such that (1) H is a Hilbert space, (2) (R, H) is a G -representation, and (3) $R: G \rightarrow \text{Hom}(H, H)$ has the additional property that $R(x^{-1}) = [R(x)]^*$ for all $x \in G$, where $[R(x)]^*$ denotes the Hilbert space adjoint of $R(x)$.

27.22 Definition. Let G be a k -group. Then a $M(G)$ *-representation will be an ordered pair (R, H) such that (1) H is a Hilbert space, (2) (R, H) is a $M(G)$ -representation, and (3) $R: M(G) \rightarrow \text{Hom}(H, H)$ has the

additional property that $R(\mu^*) = [R(\mu)]^*$ for all $\mu \in M(G)$, where $[R(\mu)]^*$ denotes the Hilbert space adjoint of $R(\mu)$.

27.23 Note. Since Hilbert spaces are barrelled and p -complete, one can use proposition 27.16 to show that my definitions of G -unitary representation and $M(G)$ $*$ -representations agree with the standard notions of "strongly continuous unitary representation" and "strongly continuous $*$ -representation" respectively.

27.24 Remark. In the definitions above and for the remainder of this section, I shall regard the category of Hilbert spaces (and continuous linear maps) as though it were a full subcategory of the category of p -reflexive spaces (and continuous linear maps). Of course, this is not the case since Hilbert spaces have more structure (i.e. an inner product) than p -reflexive spaces. What is true is that there is a full, faithful "forgetful" functor from the category of Hilbert spaces (and continuous linear maps) to the category of p -reflexive spaces (and continuous linear maps). Thus in order to make the above definitions strictly correct and in order to make some of what follows strictly correct, forgetful functors should be sprinkled appropriately throughout. I have chosen to be sloppy in this regard in order not to obscure (any more than necessary) what is actually going on.

I must confess that earlier I glossed over similar

difficulties. Specifically, I have not been distinguishing between metrizable spaces and metric spaces in many cases, similarly between normable spaces and normed spaces.

27.25 Definition. Let G be a k -group. Define $\text{Rep}^*(G)$ to be the full subcategory of $\text{Rep}(G)$ whose objects are G -unitary representations (cf. 27.24).

27.26 Definition. Let G be a k -group. Define $\text{Rep}^*(M(G))$ to be the full subcategory of $\text{Rep}(M(G))$ whose objects are $M(G)$ $*$ -representations (cf. 27.24).

27.27 Theorem. Let G be a k -group. Then the categories $\text{Rep}^*(G)$ and $\text{Rep}^*(M(G))$ are isomorphic.

Proof: Let $A: \text{Rep}(G) \rightarrow \text{Rep}(M(G))$ and $B: \text{Rep}(M(G)) \rightarrow \text{Rep}(G)$ be the functors defined in the proof of 27.19. Since I proved in 27.19 that $A \circ B = \text{id}_{\text{Rep}(M(G))}$ and $B \circ A = \text{id}_{\text{Rep}(G)}$, and since $\text{Rep}^*(M(G))$ and $\text{Rep}^*(G)$ are full subcategories of $\text{Rep}(M(G))$ and $\text{Rep}(G)$ respectively (cf. 27.24); I will be done if I can prove

- i) if (R, V) is an object of $\text{Rep}^*(G)$, then $A(R, V)$ is an object of $\text{Rep}^*(M(G))$; and
- ii) if (R, V) is an object of $\text{Rep}^*(M(G))$, then $B(R, V)$ is an object of $\text{Rep}^*(G)$.

Now ii) is trivial, since $B(R, V) = (R \circ \varepsilon, V)$ where $\varepsilon: G \rightarrow M(G)$ is the canonical map. But by 27.6, we have $\varepsilon(x^{-1}) = [\varepsilon(x)]^*$ for all $x \in G$. So let us prove i).

Let (R, V) be an object in $\text{Rep}^*(G)$, i.e. R is a continuous map from G to $\text{Hom}(V, V)$, where V is a Hilbert space, such that

- a) $R(xy) = R(x) \circ R(y)$ for all $x, y \in G$;
- b) $R(e) = I$; and
- c) $R(x^{-1}) = [R(x)]^*$ for all $x \in G$.

Let $v, w \in V$. Define $\alpha_1: \text{Hom}(V, V) \rightarrow \mathbb{K}$ and $\alpha_2: \text{Hom}(V, V) \rightarrow \mathbb{K}$ by $\alpha_1(T) = \langle T(w), v \rangle$ and $\alpha_2(T) = \langle T(v), w \rangle$, where $\langle -, - \rangle$ denotes the inner product of V . Notice that α_1 and α_2 are both continuous linear operators.

By definition, $A(R, V) = (S, V)$ where $S: M(G) \rightarrow \text{Hom}(V, V)$ is the unique continuous linear map such that

$$\begin{array}{ccc}
 G & \xrightarrow{R} & \text{Hom}(V, V) \\
 \varepsilon \downarrow & \nearrow S & \\
 M(G) & &
 \end{array}$$

commutes. Now by 10.7, we get that for $i = 1$ and $i = 2$, $(\alpha_i \circ S)(v) = v(\alpha_i \circ R)$ for all $v \in M(G)$.

Let $\mu \in M(G)$. By 27.6,
 $\mu^*(\alpha_1 \circ R) = \overline{\mu(x \mapsto (\alpha_1 \circ R)(x^{-1}))}$ where the bars denote complex conjugation. Now
 $\overline{(\alpha_1 \circ R)(x^{-1})} = \overline{\langle R(x^{-1})(w), v \rangle} = \langle v, R(x^{-1})(w) \rangle = \langle v, [R(x)]^*(w) \rangle = \langle R(x)(v), w \rangle$. So
 $\mu^*(\alpha_1 \circ R) = \overline{\mu(x \mapsto \langle R(x)(v), w \rangle)} = \overline{\mu(\alpha_2 \circ R)}$. Hence

$$\begin{aligned} \langle v, S(\mu^*)(w) \rangle &= \overline{\langle S(\mu^*)(w), v \rangle} = \overline{\alpha_1(S(\mu^*))} = \overline{(\alpha_1 \circ S)(\mu^*)} = \\ &= \overline{\mu^*(\alpha_1 \circ R)} = \mu(\alpha_2 \circ R) = (\alpha_2 \circ S)(\mu) = \alpha_2(S(\mu)) = \langle S(\mu)(v), w \rangle. \end{aligned}$$

Hence for all $v, w \in V$ and $\mu \in M(G)$,

$$\langle v, S(\mu^*)(w) \rangle = \langle S(\mu)(v), w \rangle. \text{ Thus } S(\mu^*) = [S(\mu)]^*.$$

Hence $A(R, V) = (S, V)$ is a $M(G)$ *-representation,
i.e. $A(R, V)$ is an object of $\text{Rep}^*(M(G))$. QED on i)

QED

Appendix A - Elementary facts from category theory.

A1 Lemma. If \mathcal{O} , \mathcal{O}' , \mathcal{B} , and \mathcal{B}' are arbitrary categories; $R: \mathcal{O}' \rightarrow \mathcal{O}$, $S: \mathcal{B} \rightarrow \mathcal{B}'$, $F: \mathcal{O} \rightarrow \mathcal{B}$ and $G: \mathcal{O} \rightarrow \mathcal{B}$ are functors; and $\phi: F \rightarrow G$ is a natural transformation (respectively, isomorphism); then $\phi': S \circ F \circ R \rightarrow S \circ G \circ R$ defined by $A' \mapsto S(\phi(R(A')))$ is a natural transformation (respectively, isomorphism).

A2 Lemma. If \mathcal{O} and \mathcal{B} are arbitrary categories; F , G , and H are functors from \mathcal{O} to \mathcal{B} ; and $\phi: F \rightarrow G$ and $\psi: G \rightarrow H$ are natural transformations (resp., isomorphisms); then $\psi \circ \phi: F \rightarrow H$ is a natural transformation (resp., isomorphism).

A3 Lemma. Suppose \mathcal{O} , \mathcal{B} , and \mathcal{C} are arbitrary categories and $F: \mathcal{O} \rightarrow \mathcal{B}$, $G: \mathcal{B} \rightarrow \mathcal{C}$, $H: \mathcal{C} \rightarrow \mathcal{B}$, and $J: \mathcal{B} \rightarrow \mathcal{O}$ are functors such that F is an adjoint of J and G is an adjoint of H . Then $G \circ F$ is an adjoint of $J \circ H$.

Proof of A1: Suppose $t \in \text{Mor}_{\mathcal{O}'}(L, M)$. Then $R(t) \in \text{Mor}_{\mathcal{O}}(R(L), R(M))$. So the following commutes:

$$\begin{array}{ccc}
 F(R(L)) & \xrightarrow{F(R(t))} & F(R(M)) \\
 \phi(R(L)) \downarrow & & \downarrow \phi(R(M)) \\
 G(R(L)) & \xrightarrow{G(R(t))} & G(R(M))
 \end{array}$$

Thus the following commutes:

$$\begin{array}{ccc}
 S(F(R(L))) & \xrightarrow{S(F(R(t)))} & S(F(R(M))) \\
 \downarrow & & \downarrow \\
 S(\phi(R(L))) & & S(\phi(R(M))) \\
 \downarrow & & \downarrow \\
 S(G(R(L))) & \xrightarrow{S(G(R(t)))} & S(G(R(M)))
 \end{array}$$

The only other thing to note (other than the definitions) is that functors take isomorphisms to isomorphisms.

QED on A1

Proof of A2: Let $f \in \text{Mor}_{\mathcal{O}}(A, A')$. Then consider the following:

$$\begin{array}{ccc}
 F(A) & \xrightarrow{F(f)} & F(A') \\
 \phi(A) \downarrow & [1] & \downarrow \phi(A') \\
 G(A) & \xrightarrow{G(f)} & G(A') \\
 \psi(A) \downarrow & [2] & \downarrow \psi(A') \\
 H(A) & \xrightarrow{H(f)} & H(A')
 \end{array}$$

[1] commutes because $\phi: F \rightarrow G$ and [2] commutes because $\psi: G \rightarrow H$. Thus the large diagram commutes. QED

Proof of A3: Suppose $R, S: \mathcal{O}^{\text{OP}} \times \mathcal{B} \rightarrow \text{Sets}$ and $T, U: \mathcal{B}^{\text{OP}} \times \mathcal{C} \rightarrow \text{Sets}$ are defined by

$$R(A, B) = \text{Mor}_{\mathcal{A}}(F(A), B), \quad S(A, B) = \text{Mor}_{\mathcal{O}}(A, J(B)),$$

$$T(B, C) = \text{Mor}_{\mathcal{C}}(G(B), C), \quad \text{and} \quad U(B, C) = \text{Mor}_{\mathcal{B}}(B, H(C)).$$

Suppose $\phi: R \rightarrow S$ and $\psi: T \rightarrow U$ are natural isomorphisms. Notice that $R \circ (\text{id} \times H)$ and $U \circ (F^{\text{OP}} \times \text{id})$

[where id denotes the appropriate identity functor] are equal as functors from $\mathcal{O}^{\text{OP}} \times \mathcal{C} \rightarrow \text{Sets}$. Also

$R \circ (\text{id} \times H) \cong S \circ (\text{id} \times H)$ and $T \circ (F^{\text{OP}} \times \text{id}) \cong$

$U \circ (F^{\text{OP}} \times \text{id})$ by A1. Thus $T \circ (F^{\text{OP}} \times \text{id}) \cong$

$S \circ (\text{id} \times H)$ by A2. But

$[T \circ (F^{\text{OP}} \times \text{id})](A, C) = \text{Mor}_{\mathcal{C}}(G(F(A)), C)$ and

$[S \circ (\text{id} \times H)](A, C) = \text{Mor}_{\mathcal{O}}(A, J(H(C)))$. Hence $G \circ F$

is an adjoint of $J \circ H$.

QED on A3

A4 Lemma. Let \mathcal{C} and \mathcal{D} be arbitrary categories, and let $R: \mathcal{C} \rightarrow \mathcal{D}$ and $S: \mathcal{C} \rightarrow \mathcal{D}$ be functors. If $R \cong S$ and R has an adjoint T , then T is an adjoint of S also.

Proof: Suppose $\lambda: R \rightarrow S$ is a natural isomorphism. Then $\lambda': \text{id} \times R \rightarrow \text{id} \times S$ defined by $\lambda'(D, C) = 1_D \times \lambda(C)$ is a natural isomorphism. So $\text{Mor}_{\mathcal{D}}(\text{id} \times R)$ is naturally isomorphic to $\text{Mor}_{\mathcal{D}}(\text{id} \times S)$ by A1. But $\text{Mor}_{\mathcal{C}}(T^{\text{OP}} \times \text{id})$ is naturally isomorphic to $\text{Mor}_{\mathcal{D}}(\text{id} \times R)$ since T is an adjoint of R . So by A2, $\text{Mor}_{\mathcal{C}}(T^{\text{OP}} \times \text{id})$ is naturally isomorphic to $\text{Mor}_{\mathcal{D}}(\text{id} \times S)$, i.e. T is an adjoint of S . QED

A5 Lemma. Suppose \mathcal{O} and \mathcal{B} are arbitrary categories and F and G are functors from \mathcal{O} to \mathcal{B} . Define functors R and S from $\mathcal{O}^{\text{OP}} \times \mathcal{B}$ to Sets by $R(A, B) = \text{Mor}_{\mathcal{B}}(G(A), B)$ and $S(A, B) = \text{Mor}_{\mathcal{B}}(F(A), B)$. Suppose $\phi: R \rightarrow S$ is a natural transformation. Then $\psi: F \rightarrow G$ defined by $\psi(A) = \phi(A, G(A))(1_{G(A)})$ is a

natural transformation, and ψ is a natural isomorphism if ϕ is.

Proof: Suppose $A, A' \in \text{ob } \mathcal{O}$ and $f \in \text{Mor}_{\mathcal{O}}(A, A')$. So since ϕ is natural, the following two diagrams commute:

$$\begin{array}{ccc}
 R(A, G(A)) & \xrightarrow{R(l_A, G(f))} & R(A, G(A')) \\
 \downarrow \phi(A, G(A)) & & \downarrow \phi(A, G(A')) \\
 S(A, G(A)) & \xrightarrow{S(l_A, G(f))} & S(A, G(A')) \\
 \\
 R(A', G(A')) & \xrightarrow{R(f, l_{G(A')})} & R(A, G(A')) \\
 \downarrow \phi(A', G(A')) & & \downarrow \phi(A, G(A')) \\
 S(A', G(A')) & \xrightarrow{S(f, l_{G(A')})} & S(A, G(A'))
 \end{array}$$

Now $l_{G(A)} \in R(A, G(A))$ and $l_{G(A')} \in R(A', G(A'))$. So $R(l_A, G(f))(l_{G(A)}) = G(f) \circ l_{G(A)} \circ l_{G(A)} = G(f)$ and $R(f, l_{G(A')})(l_{G(A')}) = l_{G(A')} \circ l_{G(A')} \circ G(f) = G(f)$. So

$$\begin{aligned}
 S(l_A, G(f))[\phi(A, G(A))(l_{G(A)})] &= \phi(A, G(A'))[R(l_A, G(f))(l_{G(A)})] \\
 &= \phi(A, G(A'))[R(f, l_{G(A')})(l_{G(A')})] \\
 &= S(f, l_{G(A')})[\phi(A', G(A'))(l_{G(A')})].
 \end{aligned}$$

Thus $G(f) \circ \psi(A) = \psi(A') \circ F(f)$. So ψ is a natural transformation from F to G .

Now suppose ϕ is a natural isomorphism. I claim if $A \in \text{ob } \mathcal{O}$, then $\psi(A)^{-1} = \phi(A, F(A))^{-1}(l_{F(A)})$.

Let $g = \phi(A, F(A))^{-1}(l_{F(A)})$. Then the following two diagrams commute:

$$\begin{array}{ccc}
 R(A, F(A)) & \xrightarrow{R(l_A, \psi(A))} & R(A, G(A)) \\
 \downarrow \phi(A, F(A)) & & \downarrow \phi(A, G(A)) \\
 S(A, F(A)) & \xrightarrow{S(l_A, \psi(A))} & S(A, G(A))
 \end{array}$$

$$\begin{array}{ccc}
 R(A, G(A)) & \xrightarrow{R(l_A, g)} & R(A, F(A)) \\
 \downarrow \phi(A, G(A)) & & \downarrow \phi(A, F(A)) \\
 S(A, G(A)) & \xrightarrow{S(l_A, g)} & S(A, F(A))
 \end{array}$$

So using the top diagram, we find that

$$\phi(A, G(A)) [R(l_A, \psi(A))(g)] = S(l_A, \psi(A)) [\phi(A, F(A))(g)], \text{ i.e.}$$

$$\begin{aligned}
 \phi(A, G(A)) [\psi(A) \circ g] &= \psi(A) \circ [\phi(A, F(A))(g)] = \psi(A) \circ l_{F(A)} \\
 &= \psi(A) \\
 &= \phi(A, G(A))(l_{G(A)})
 \end{aligned}$$

Thus by applying $\phi(A, G(A))^{-1}$ to both sides of the last equation, we get $\psi(A) \circ g = l_{G(A)}$.

From the bottom diagram,

$$\phi(A, F(A)) [R(l_A, g)(l_{G(A)})] = S(l_A, g) [\phi(A, G(A))(l_{G(A)})],$$

i.e. $l_{F(A)} = \phi(A, F(A))[g] = g \circ \psi(A)$. Hence $g = \psi(A)^{-1}$.

Hence ψ is a natural isomorphism. QED

A6 Lemma. Let \mathcal{C} be an arbitrary category and let \mathcal{D} be a full coreflective subcategory of \mathcal{C} . Let $S: \mathcal{C} \rightarrow \mathcal{D}$ be the coreflector and $\mu: S \rightarrow \text{id}_{\mathcal{C}}$ be the canonical

natural transformation. Let \mathcal{O} be a diagram scheme (i.e. a small category) and $F: \mathcal{O} \rightarrow \mathcal{D}$ be a diagram in \mathcal{D} (i.e. a functor). If $(L, \ell_A)_{A \in \text{ob } \mathcal{O}}$ is a limit of F in \mathcal{C} , then $(S(L), \ell_A \circ \mu_L)_{A \in \text{ob } \mathcal{O}}$ is a limit of F in \mathcal{D} .

Note: See [18] for the terminology.

Proof: 1) Define $\bar{\ell}_A = \ell_A \circ \mu_L$ for all $A \in \text{ob } \mathcal{O}$. Then if A and $A' \in \text{ob } \mathcal{O}$ and $f \in \text{Mor}_{\mathcal{O}}(A, A')$, we have that diagram [1] below commutes, since diagram [2] does.

$$\begin{array}{ccc}
 \text{[1]} & \begin{array}{c} S(L) \\ \swarrow \bar{\ell}_A \quad \searrow \bar{\ell}_{A'} \\ F(A) \xrightarrow{F(f)} F(A') \end{array} & \text{[2]} & \begin{array}{c} L \\ \swarrow \ell_A \quad \searrow \ell_{A'} \\ F(A) \xrightarrow{F(f)} F(A') \end{array}
 \end{array}$$

2) Suppose $L' \in \text{ob } \mathcal{D}$ and $\ell'_A: L' \rightarrow F(A)$ are morphisms for all $A \in \text{ob } \mathcal{O}$ with the property that if A and $A' \in \text{ob } \mathcal{O}$ and $f \in \text{Mor}_{\mathcal{O}}(A, A')$, we have that

$$\begin{array}{ccc}
 & L' & \\
 \ell'_A \swarrow & & \searrow \ell'_{A'} \\
 F(A) & \xrightarrow{F(f)} & F(A')
 \end{array}$$

commutes. Then there exists a unique $p: L' \rightarrow L$ such that for all $A \in \text{ob } \mathcal{O}$, the following diagram commutes:

$$\begin{array}{ccc}
 L' & \xrightarrow{p} & L \\
 \ell'_A \searrow & & \swarrow \ell_A \\
 & F(A) &
 \end{array}$$

This is true because (L, ϱ_A) is a limit.

By the coreflective property, there exists a unique $\bar{p}: L' \rightarrow S(L)$ such that

$$\begin{array}{ccc} & & S(L) \\ & \nearrow \bar{p} & \downarrow \mu_L \\ L' & & L \\ & \searrow p & \end{array}$$

commutes. So for all $A \in \text{ob}\mathcal{O}$, the following diagram commutes:

$$\begin{array}{ccc} L' & \xrightarrow{\bar{p}} & S(L) \\ \varrho'_A \downarrow & & \downarrow \mu_L \\ F(A) & \xleftarrow{\varrho_A} & L \end{array}$$

So there exists a $\bar{p}: L' \rightarrow S(L)$ such that $\bar{\varrho}_A \circ \bar{p} = \varrho'_A$ for all $A \in \text{ob}\mathcal{O}$.

3) Next suppose that $k: L' \rightarrow S(L)$ such that

$$\begin{array}{ccc} L' & \xrightarrow{k} & S(L) \\ \varrho'_A \downarrow & & \downarrow \mu_L \\ F(A) & \xleftarrow{\varrho_A} & L \end{array}$$

commutes for all $A \in \text{ob}\mathcal{O}$. Then $\mu_L \circ k = p$ because (L, ϱ_A) is a limit. But \bar{p} was the unique from L' to $S(L)$ such that $p = \mu_L \circ \bar{p}$. Hence $k = \bar{p}$. So there exists at most one $\bar{p}: L' \rightarrow S(L)$ such that $\bar{\varrho}_A \circ \bar{p} = \varrho'_A$ for all $A \in \text{ob}\mathcal{O}$.

4) $S(L) \in \text{ob} \mathcal{D}$ because S is a coreflector, and $\lambda_A \circ \mu_L = \bar{\lambda}_A \in \text{Mor}_{\mathcal{D}}(S(L), F(A))$ for all $A \in \text{ob} \mathcal{A}$ since the category \mathcal{D} is full in \mathcal{C} and since $S(L)$ and $F(A)$ are in $\text{ob} \mathcal{D}$. Similarly note that in 2), $\bar{p} \in \text{Mor}_{\mathcal{D}}$.

Hence $(S(L), \bar{\lambda}_A)_{A \in \text{ob} \mathcal{A}}$ is a limit of F in \mathcal{D} .

QED

A7 Corollary. If \mathcal{C} is a complete category and \mathcal{D} is a full coreflective subcategory of \mathcal{C} , then \mathcal{D} is complete also; and sloppily speaking, we find a limit in \mathcal{D} by first taking the limit in \mathcal{C} and then taking the coreflection of that limit in \mathcal{D} .

Proof: Limits exist in \mathcal{C} since it is complete; so just apply A6. QED

Appendix B - Elementary categorical properties of
locally convex spaces.

Let \mathbb{K} denote the scalar field which will be either the complex numbers or the real numbers.

Let \mathcal{C}_n denote the category of locally convex topological vector spaces over \mathbb{K} and continuous linear maps.

Let \mathcal{C} denote the full subcategory of \mathcal{C}_n consisting of locally convex Hausdorff spaces.

It is well-known that

B1 Proposition. \mathcal{C} is a epireflective subcategory of \mathcal{C}_n .

Proof. Let E be a locally convex space. Let $\{0\}^-$ denote the closure of $\{0\}$ in E . Then $j: E \rightarrow E/\{0\}^-$ is the reflection of E in \mathcal{C} as is easily verified. QED

It is equally well-known that

B2 Proposition. The categories \mathcal{C} and \mathcal{C}_n are both complete and cocomplete; i.e. limits and colimits exist in both categories (cf. [3]).

For example:

1) in both \mathcal{C} and \mathcal{C}_n , if $f: E \rightarrow F$ is a mor-

phism, then the kernel of f is $i: f^{-1}(\{0\}) \rightarrow E$ where i is the inclusion map and $f^{-1}(\{0\})$ is given the relative topology from E ;

2) in \mathcal{C} , if $f: E \rightarrow F$ is a morphism, then the cokernel of f is $j: F \rightarrow F/[f(E)]^-$ where $F/[f(E)]^-$ has the quotient topology and j is the canonical projection; and

3) in \mathcal{C}_n , if $f: E \rightarrow F$ is a morphism, then the cokernel of f is $j: F \rightarrow F/f(E)$ where $F/f(E)$ has the quotient topology and j is the canonical projection.

B3 Corollary. The \mathcal{C} -limit of a diagram in \mathcal{C} agrees with the \mathcal{C}_n -limit of that same diagram.

Proof: \mathcal{C} is an epireflective subcategory of \mathcal{C}_n . Thus the inclusion functor preserves limits. QED

B4 Proposition. Let \mathcal{E} be a full subcategory of \mathcal{C}_n with the property that if $E \in \text{ob } \mathcal{E}$ and A is a subspace of E (not necessarily closed), then $E/A \in \text{ob } \mathcal{E}$. Suppose E and $F \in \text{ob } \mathcal{E}$ and $f: E \rightarrow F$ is an \mathcal{E} -morphism. Then f is epic iff f is surjective.

Proof: If f is surjective, then f is obviously epic.

Suppose f is epic. Let $r: F \rightarrow F/f(E)$ be the canonical projection and $s: F \rightarrow F/f(E)$ be the zero map. $F/f(E) \in \text{ob } \mathcal{E}$ by assumption, and r and s are \mathcal{E} -morphisms since \mathcal{E} is a full subcategory of \mathcal{C}_n . Now $s \circ f = r \circ f$. So $s = r$, since f is epic. Let

$x \in F$. Then $f(E) = 0 + f(E) = s(x) = r(x) = x + f(E)$.
So $x \in f(E)$. Hence $F = f(E)$, i.e. f is surjective.

QED

B5 Proposition. Let \mathcal{E} be a full subcategory of \mathcal{C} such that $\mathbb{K} \in \text{ob}\mathcal{E}$. Suppose E and $F \in \text{ob}\mathcal{E}$ and $f: E \rightarrow F$ is an \mathcal{E} -morphism. Then f is epic iff f has a dense image.

Proof: Suppose f has a dense image. Let $G \in \text{ob}\mathcal{E}$, $r: F \rightarrow G$, and $s: F \rightarrow G$ such that $r \circ f = s \circ f$. Now since $G \in \text{ob}\mathcal{E}$, G is Hausdorff. Hence $\{x \in F : r(x) = s(x)\}$ is a closed subspace of F which contains the image of f . Thus $\{x \in F : r(x) = s(x)\}$ contains the closure of the image of f . Hence $r = s$. So f is epic.

Suppose f is epic. Let ϕ be any linear functional on F such that $\phi(f(x)) = 0$ for all $x \in E$. Since $\mathbb{K} \in \text{ob}\mathcal{E}$ and \mathcal{E} is a full subcategory of \mathcal{C} , ϕ is an \mathcal{E} -morphism. Similarly the zero map from F to \mathbb{K} is an \mathcal{E} -morphism. So $0 \circ f = \phi \circ f$. Hence since f is epic, $0 = \phi$. By the Hahn-Banach theorem, this implies that the image of f is dense in F . QED

B6 Proposition. Let \mathcal{E} be a full subcategory of \mathcal{C}_n such that $\mathbb{K} \in \text{ob}\mathcal{E}$. Suppose E and F are elements of $\text{ob}\mathcal{E}$ and $f: E \rightarrow F$ is an \mathcal{E} -morphism. Then f is monic iff f is injective.

Proof: If f is injective, then f is trivially

monic. Suppose f is monic. Let $x \in E$ such that $f(x) = 0$. Let $r: K \rightarrow E$ denote the map defined by $t \mapsto tx$. r is a \mathcal{C}_n -morphism and hence a \mathcal{E} -morphism, since \mathcal{E} is a full subcategory of \mathcal{C}_n . Similarly the zero map is also a \mathcal{E} -morphism. Now for all $t \in K$, $(f \circ 0)(t) = f(0(t)) = f(0) = 0 = t0 = tf(x) = f(tx) = f(r(t)) = (f \circ r)(t)$. So $f \circ 0 = f \circ r$. Thus $0 = r$, since f is monic. So $0 = r(1) = x$. Hence f is injective. QED

B7 Proposition. Let A and $B \in \mathcal{C}$ and $f: A \rightarrow B$ be a \mathcal{C} -morphism. Then

- 1) f is the \mathcal{C} -kernel of some morphism iff $f(A)$ is closed in B and f is a \mathcal{C} -isomorphism onto $f(A)$; and
- 2) The following three statements are equivalent:
 - a) f is the \mathcal{C} -cokernel of some morphism.
 - b) f induces an isomorphism from $A/\ker f$ onto B .
 - c) f is a surjective, open mapping.

Proof:

1) (\Leftarrow) Suppose $f(A)$ is closed in B and f is an isomorphism onto $f(A)$. Let $p: B \rightarrow B/f(A)$ be the canonical projection. I claim that f is the kernel of p .

First note that $p \circ f = 0$.

Suppose $g: X \rightarrow B$ and $p \circ g = 0$. Then $g(X) \subset \ker p = f(A)$. Let $r: X \rightarrow A$ be defined by $r(x) = f^{-1}(g(x))$. r is continuous, linear, and

$f \circ r = g$. Suppose $r': X \rightarrow A$ and $f \circ r' = g$. Then $f \circ r' = f \circ r$. But f is injective, hence monic. So $r' = r$. Thus f is a kernel of p .

(\Rightarrow) Let $D =$ the closure of $f(A)$. Suppose $p: B \rightarrow C$ and f is the kernel of p . Then $p \circ f = 0$. So $f(A) \subset \ker p$. Thus $D \subset \ker p$. Give D the relative topology from B and let $i: D \rightarrow B$ be the inclusion map. Then $p \circ i = 0$. So there exists a unique map $r: D \rightarrow A$ such that $f \circ r = i$. Let $d \in D$. Then $f(r(d)) = i(d) = d$. So $D \subset f(A)$. Thus $f(A)$ is closed in B , i.e. $f(A) = D$. Define $h: A \rightarrow D$ by $h(x) = f(x)$. Then $i \circ h = f$. So we have $f \circ r = i$ and $i \circ h = f$. Hence $i \circ h \circ r = i = i \circ l_D$ and $f \circ r \circ h = f = f \circ l_A$. But f is monic because it's a kernel and i is monic because it's injective. So $h \circ r = l_D$ and $r \circ h = l_A$. So f is an isomorphism onto its range. QED on 1)

2) ($b \Rightarrow a$) Suppose f induces an isomorphism from $A/\ker f$ to B . Let $D = \ker f$. Let D have the relative topology and let $p: D \rightarrow A$ be the inclusion map. I claim f is a cokernel of p .

First note that $f \circ p = 0$.

Secondly suppose $g: A \rightarrow X$ and $g \circ p = 0$. Then $p(D) \subset \ker g$, i.e. $\ker f \subset \ker g$. So there exists a continuous linear map $\bar{g}: A/\ker f \rightarrow X$ such that

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ q \downarrow & \nearrow & \\ A/\ker f & \xrightarrow{\bar{g}} & X \end{array}$$

commutes, where q is the canonical projection of A onto $A/\ker f$. Let $\bar{f}: A/\ker f \rightarrow B$ be such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ q \downarrow & \nearrow \bar{f} & \\ A/\ker f & & \end{array}$$

commutes. \bar{f} is an isomorphism by assumption. Let $r = \bar{g} \circ \bar{f}^{-1}$. We have that $r \in \text{Mor}_{\mathcal{C}}(B, X)$ and $r \circ f = g$, since $r \circ f = r \circ \bar{f} \circ q = (\bar{g} \circ \bar{f}^{-1}) \circ \bar{f} \circ q = \bar{g} \circ q = g$. Hence we have existence.

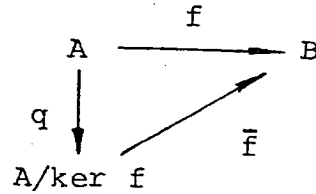
As for uniqueness, suppose that $r': B \rightarrow X$ and $r' \circ f = g$. Then $r' \circ f = r \circ f$. But f is surjective since \bar{f} and q are, thus f is epic. So $r' = r$.

Hence f is a cokernel of p . QED on $b \Rightarrow a$

(a \Rightarrow b) Suppose f is a cokernel of a morphism $p: C \rightarrow A$. Then $f \circ p = 0$, so $p(C) \subset \ker f$. Let $q: A \rightarrow A/\ker f$ be the canonical map. Then $q \circ p = 0$. So there exists a map $r: B \rightarrow A/\ker f$ such that $r \circ f = q$. Let $\bar{f}: A/\ker f \rightarrow B$ be the unique map such that $\bar{f} \circ q = f$. Thus $\bar{f} \circ q = f$ and $r \circ f = q$. Hence $r \circ \bar{f} \circ q = q = 1_{A/\ker f} \circ q$ and $\bar{f} \circ r \circ f = f = 1_B \circ f$. But cokernels are epic and q is surjective and hence epic. So $\bar{f} \circ r = 1_B$ and $r \circ \bar{f} = 1_{A/\ker f}$. So \bar{f} is an isomorphism. QED on $a \Rightarrow b$

(b \Rightarrow c) This is trivial since $\bar{f} \circ q = f$ and since q is open and surjective, where q is the projection map. QED on $b \Rightarrow c$

(c**⇒**b) We know that



commutes, and that \bar{f} is injective and continuous. f being surjective implies that \bar{f} is surjective. So \bar{f}^{-1} exists as a linear function, we need only check that it is continuous.

Let $L \subset A/\ker f$ be an open neighborhood of zero. Then $\bar{f}(L) \supset f(q^{-1}(L))$ since the above diagram commutes. So $L \supset \bar{f}^{-1}(f(q^{-1}(L)))$. But $f(q^{-1}(L))$ is an open neighborhood of zero, since f is open and q is continuous. So \bar{f} is continuous. QED on c**⇒**b

QED

Appendix C - Elementary properties of k-spaces.

C1 Definition. A topological space X is a k-space provided (1) X is Hausdorff, and (2) C is closed in X iff $C \cap K$ is closed in X for all compact subsets K of X .

C2 Theorem. If X is a k-space, Y is any topological space, and $f: X \rightarrow Y$ is any function, then f is continuous iff $f|_K$ is continuous for all compact subsets K of X . QED

C3 Theorem. The category \mathcal{J} of k-spaces and continuous maps is a full coreflective subcategory of the category \mathcal{J}_0 of Hausdorff topological spaces and continuous maps. Moreover, if $X \in \text{obj } \mathcal{J}_0$ and $\ell: X_K \rightarrow X$ is its coreflection in the category \mathcal{J} , then ℓ is bijective.

Proof: Let (X, \mathcal{J}) be a Hausdorff topological space. Let $\mathcal{J}' = \{C \subset X : C \cap K \text{ is } \mathcal{J}\text{-closed } \forall K \text{ compact in } X\}$.

Claim: \mathcal{J}' forms the set of closed sets for a topology on X .

Proof: Trivial.

Let \mathcal{J}^* be the topology induced by \mathcal{J}' , i.e.

$\mathcal{J}^* = \{X \setminus S : S \in \mathcal{J}'\}$. Note that $\mathcal{J} \subset \mathcal{J}^*$. Hence \mathcal{J}^* is a Hausdorff topology.

Claim: If $K \subset X$ is \mathcal{J} -compact, then the relative

topologies on K induced by (X, \mathcal{S}) and (X, \mathcal{S}^*) agree; in particular, K is also \mathcal{S}^* -compact.

Proof of claim: Let $\mathcal{S}|_K$ and $\mathcal{S}^*|_K$ denote the two relative topologies in question. We would like to prove that $\mathcal{S}|_K = \mathcal{S}^*|_K$. Since $\mathcal{S} \subset \mathcal{S}^*$, we know that $\mathcal{S}|_K \subset \mathcal{S}^*|_K$.

Suppose $A \subset K$ is $\mathcal{S}^*|_K$ -closed. Then there exists a \mathcal{S}^* -closed subset $R \subset X$ such that $A = K \cap R$. Since R is \mathcal{S}^* -closed, $R \in \mathcal{S}'$. Hence A is \mathcal{S} -closed in X . But $A = A \cap K$. So A is $\mathcal{S}|_K$ -closed. Hence A $\mathcal{S}^*|_K$ closed implies A is $\mathcal{S}|_K$ closed. So $\mathcal{S}^*|_K \subset \mathcal{S}|_K$. Hence $\mathcal{S}^*|_K = \mathcal{S}|_K$. QED on claim

Claim: (X, \mathcal{S}^*) is a k -space.

Proof: We already know that (X, \mathcal{S}^*) is Hausdorff. Suppose $L \subset X$ and $L \cap K$ is \mathcal{S}^* closed for all \mathcal{S}^* compact sets $K \subset X$. Let K be a \mathcal{S} compact set. Then by the last claim K is \mathcal{S}^* compact. So $K \cap L$ is \mathcal{S}^* closed. Hence $K \cap L \in \mathcal{S}'$. So $(K \cap L) \cap K = K \cap L$ is \mathcal{S} closed (recall K is \mathcal{S} compact). So for all $K \subset X$, if K is \mathcal{S} -compact, then $K \cap L$ is \mathcal{S} closed. This implies $L \in \mathcal{S}'$. So L is closed for the topology \mathcal{S}^* . Hence (X, \mathcal{S}^*) is a k -space. QED on claim

So I will be done if I can prove the following

Claim: Let $j: (X, \mathcal{S}^*) \rightarrow (X, \mathcal{S})$ be the identity map (it is continuous since $\mathcal{S}^* \supset \mathcal{S}$). Let Y be a k -space $\overset{\text{and}}{\wedge} f: Y \rightarrow (X, \mathcal{S})$ be a continuous map. Then $f: Y \rightarrow (X, \mathcal{S}^*)$ is actually continuous.

Proof of claim: Let R be \mathcal{J}^* closed in X . It suffices to show that $f^{-1}(R)$ is closed in Y . Since Y is a k -space, $f^{-1}(R)$ is closed iff $K \cap f^{-1}(R)$ is closed in Y for all K compact in Y .

Let $K \subset Y$ be compact. Then $f(K)$ is \mathcal{J} -compact in X . Since R is \mathcal{J}^* closed, $R \in \mathcal{J}'$. So $f(K) \cap R$ is \mathcal{J} closed. Thus $f^{-1}(f(K) \cap R) = f^{-1}(f(K)) \cap f^{-1}(R)$ is closed in Y . So $K \cap (f^{-1}(f(K)) \cap f^{-1}(R)) = K \cap f^{-1}(R)$ is closed in Y since K is closed in Y . Thus $f^{-1}(R)$ is closed in Y .

So $f: Y \rightarrow (X, \mathcal{J}^*)$ is continuous. QED on claim

QED

C4 Corollary to proof. Let \mathcal{J} and \mathcal{J}_0 be as in the statement of C3. Let $X \in \text{ob } \mathcal{J}_0$ and X_k be its coreflection in \mathcal{J} . Let K be a compact subset of X . Then the topologies on K induced by X and by X_k agree; so in particular, K is compact in X_k . QED

See C13 for a related result.

C5 Theorem. Let I be an index set. Suppose X is a set; and for all $\alpha \in I$, Y_α is a k -space and $f_\alpha: Y_\alpha \rightarrow X$ is a function. Let \mathcal{Q} be the finest topology on X making each f_α continuous. Then if (X, \mathcal{Q}) is Hausdorff, (X, \mathcal{Q}) is a k -space.

Proof: Let $R \subset X$ be such that for all $K \subset X$ which are \mathcal{Q} -compact, $K \cap R$ is \mathcal{Q} -closed. We would

like to conclude that R is Q -closed. Now R is Q -closed iff $f_\alpha^{-1}(R)$ is closed in Y_α for all $\alpha \in I$. Let $\alpha \in I$ and K be compact in Y_α . Then $f_\alpha(K)$ is Q -compact, since $f_\alpha: Y_\alpha \rightarrow (X, Q)$ is continuous. So $f_\alpha(K) \cap R$ is Q -closed. Hence $f_\alpha^{-1}(f_\alpha(K) \cap R) = f_\alpha^{-1}(f_\alpha(K)) \cap f_\alpha^{-1}(R)$ is closed in Y_α . Now K is closed in Y_α , so $K \cap (f_\alpha^{-1}(f_\alpha(K)) \cap f_\alpha^{-1}(R)) = K \cap f_\alpha^{-1}(R)$ is closed in Y_α . Thus $f_\alpha^{-1}(R)$ is closed in Y_α , since Y_α is a k -space. But this is true for all $\alpha \in I$. So R is Q -closed in X . Hence (X, Q) is a k -space. QED

C6 Corollary. If Y_α is a k -space for all $\alpha \in I$, then the direct sum of the Y_α in the category of topological spaces and continuous maps is again a k -space.

Proof: The direct sum topology is a topology of the sort described in the last theorem. QED

C7 Corollary. Let X be a Hausdorff topological space. Let $\mathcal{K} =$ all compact subsets of X . Order \mathcal{K} as follows: $K_1 \leq K_2$ iff $K_1 \subset K_2$. If $\alpha, \beta \in \mathcal{K}$ and $\alpha \leq \beta$, let $i(\alpha, \beta): \alpha \rightarrow \beta$ be the inclusion map. Then the direct limit of the direct limit system $\{i(\alpha, \beta) : \alpha \leq \beta\}$ is a k -space.

Proof: For all $\alpha \in \mathcal{K}$, let $j_\alpha: \alpha \rightarrow X$ be the inclusion map. Let Q be the finest topology on X making each j_α continuous. Note that Q is finer than the original topology and hence is Hausdorff. It is easy to verify that $j_\alpha: \alpha \rightarrow (X, Q)$ is the direct limit

of the direct limit system $\{i(\alpha, \beta) : \alpha \leq \beta\}$. By C5,
 (X, Q) is a k -space. QED

C8 Corollary. Let everything be the same as the last
 corollary. Then if $i_\alpha: \alpha \rightarrow X$ is the inclusion map
 for all $\alpha \in K$, then i_α is the direct limit of $i(\alpha, \beta)$
 iff X is a k -space.

Proof: (\Rightarrow) By the universal property of direct
 limits, X must be homeomorphic to the space above. It
 is a k -space, hence X must be.

(\Leftarrow) This direct is a trivial consequence of C3. QED

C9 Lemma. A closed subspace of a k -space with the re-
 lative topology is a k -space. QED

C10 Theorem. The category of k -spaces and continuous
 maps is a complete category; in particular, products
 exists. For example, if for all $\alpha \in I$, X_α is a k -space,
 the the product in the category of k -spaces of the X_α
 is the coreflection in that category of the product of
 the X_α in the category of topological Hausdorff spaces.

Proof: \mathcal{J}_0 is a complete category, and \mathcal{J} is a
 full coreflective subcategory of \mathcal{J}_0 (I am using the
 notation of C3). In such a situation, \mathcal{J} is complete
 and the limits in \mathcal{J} are just the coreflection of the
 limits in \mathcal{J}_0 (cf. A7). QED

C11 Remark. There are examples of k -spaces X and Y
 whose product with the usual product topology is not a

k -space (cf. p. 139 of [45]).

C12 Remark. Pták [36] gives an example of a Hausdorff, completely regular space whose coreflection in the category of k -spaces is not completely regular. One must be on guard for this sort of undesirable behavior.

To be somewhat more detailed, Pták observes in theorem 6.17 of [36] that a Hausdorff, completely regular space T , first discussed by Novák [35], has the properties that (a) T is not a k -space; and (b) if X is a topological space and $f: T \rightarrow X$ is a function such that for all $K \subset T$ which are compact, $f|_K: K \rightarrow X$ is continuous, then $f: T \rightarrow X$ is continuous.

This being the case, the following lemma will finish this matter.

C12a Lemma. If T_k denotes the coreflection in the category of k -spaces of T , then T_k is a k -space which is not completely regular.

Proof: Let $c(T)$ and $c(T_k)$ denote the continuous real-valued functions of T and T_k respectively. Using (1) the fact that if K is a compact subset of T , then the relative topologies on K induced by T and T_k agree, and (2) property (b) of T stated above, we find that $c(T) = c(T_k)$.

Suppose T_k is completely regular. Let 0 be an

open subset of T_k . Let $x \in O$. Then there exists a function $f \in c(T_k)$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in (T \sim O)$. But we also have that $f \in c(T)$. So $f^{-1}(\{t : |t| < 1/2\})$ is open in T and $x \in f^{-1}(\{t : |t| < 1/2\}) \subset O$. This implies that O is open in T . Hence the topology of T_k is coarser than that of T . This implies that $T = T_k$. Hence T is a k -space. But by property (a) above, T is not a k -space. Contradiction. Thus T_k is not completely regular.

QED on C12a

C13 Afterthought. Let X be a Hausdorff space and X_k be its coreflection in the category of k -spaces. Then the topology of X_k is the finest topology which agrees with the topology of X on compact sets of X .

Proof: Let R be any topology which agrees with X on compact sets of X . Let C be a closed set of R . Let K be compact in X . Then $C \cap K$ is closed in K . But K is closed in X . So $C \cap K$ is closed in X . Thus C is closed in X_k . Hence the topology of X_k is finer than R . QED

Appendix D - Applications of Freyd's existence theorem for adjoint functors to the theory of locally convex spaces.

The material in this appendix is not related to anything else in this paper. This appendix was added in order to show some applications of the abstract theory of categories to interesting questions in functional analysis.

Let \mathbb{K} denote either the field of real numbers or of complex numbers.

Let \mathcal{C}_n denote the category of locally convex topological vector spaces over \mathbb{K} and continuous linear maps.

Let \mathcal{C} denote the full subcategory of \mathcal{C}_n consisting of locally convex Hausdorff spaces.

D1 Theorem. Let \mathcal{E} be a full subcategory of \mathcal{C}_n such that (1) $\mathbb{K} \in \text{ob } \mathcal{E}$, (2) if $E_\alpha \in \text{ob } \mathcal{E}$ for all $\alpha \in I$, then $\oplus\{E_\alpha : \alpha \in I\} \in \text{ob } \mathcal{E}$, and (3) if $E \in \text{ob } \mathcal{E}$ and A is a linear subspace of E (not necessarily closed), then $E/A \in \text{ob } \mathcal{E}$.

Let V be a vector space and \mathcal{J} be a locally convex topology on V . Then there exists a locally convex topology \mathcal{J}' on V such that

$$\text{a) } \mathcal{J} \subset \mathcal{J}' ;$$

- b) $(V, \mathcal{J}') \in \text{ob } \mathcal{E}$; and
- c) if $F \in \text{ob } \mathcal{E}$ and $f: F \rightarrow (V, \mathcal{J})$ is a continuous linear map, then $f: F \rightarrow (V, \mathcal{J}')$ is continuous.

D2 Remark. The categories of barrelled spaces, of quasi-barrelled spaces, and of bornological spaces each satisfy the hypothesis of D1. Note though that the result for bornological spaces is known.

Proof of D2: pp. 215, 218, and 222 of [19].

D3 Theorem. Let \mathcal{F} be a full subcategory of \mathcal{C} such that (1) $\mathbb{K} \in \text{ob } \mathcal{F}$, (2) if $F_\alpha \in \text{ob } \mathcal{F}$ for all $\alpha \in I$, then $\prod\{F_\alpha : \alpha \in I\} \in \text{ob } \mathcal{F}$, and (3) if $F \in \text{ob } \mathcal{F}$ and A is a closed linear subspace of F , then $A \in \text{ob } \mathcal{F}$.

Let $R \in \text{ob } \mathcal{C}$. Then there exists a $\hat{R} \in \text{ob } \mathcal{F}$ and a continuous linear map $i: R \rightarrow \hat{R}$ such that

- a) i is injective and has a dense image; and
- b) if $S \in \text{ob } \mathcal{F}$ and $f: R \rightarrow S$ is a continuous linear map, then there exists a unique continuous linear map $\bar{f}: \hat{R} \rightarrow S$ such that the following diagram commutes:

$$\begin{array}{ccc}
 R & \xrightarrow{f} & S \\
 \downarrow i & \nearrow \bar{f} & \\
 \hat{R} & &
 \end{array}$$

D4 Remark: The categories of Hausdorff nuclear spaces, of Hausdorff Schwartz spaces, of Hausdorff semi-Montel spaces, of semi-reflexive Hausdorff, of Hausdorff quasi-

complete spaces, and of Hausdorff complete spaces each satisfy the hypothesis of D3. Note though that the result for Hausdorff quasi-complete and Hausdorff complete spaces is known.

Proof of D4: p. 514 of [46], pp. 296 and 299 of [27], and pp. 232 and 278 of [19].

Proof of D1 and D3: First we will need two lemmas. D1a Lemma. \mathcal{C} is locally small and \mathcal{C}_n is colocally small.

Proof of D1a: Let $E \in \text{ob } \mathcal{C}$. Note that every monomorphism (i.e. injection, see B6) with domain E is equivalent to a morphism $i: A \rightarrow E$, where A is a linear subspace of E with a locally convex topology finer than the relative topology and i is the inclusion map. Thus the power class of E is actually a power set, since there are only a set of subspaces of E and since each subspace has only a set of topologies finer than the relative topology. Hence \mathcal{C} is locally small.

Let $E \in \text{ob } \mathcal{C}_n$. Note that every epimorphism (i.e. surjection, see B4) is equivalent to a morphism $p: E \rightarrow E/A$ where A is a linear subspace (not necessarily closed), p is the canonical projection, and E/A has a locally convex topology coarser than the quotient topology. Thus the copower class of E is actually a copower set, since there are at most a set of subspaces of E and since there are at most a set of topologies coarser than the quotient topology. Hence \mathcal{C}_n is colocally small. QED on D1a

D1b Lemma. \mathbb{K} is a cogenerator of \mathcal{C} and a generator of \mathcal{C}_n .

Proof of D1b: If $f: A \rightarrow B$ is a \mathcal{C} -morphism and $f \neq 0$, then there exists an $x \in A$ such that $f(x) \neq 0$. Since B is Hausdorff, by the Hahn-Banach theorem, there exists a continuous linear map $\psi: B \rightarrow \mathbb{K}$ such that $\psi(f(x)) = 1$. Hence $\psi \circ f \neq 0$. So \mathbb{K} is a cogenerator of \mathcal{C} .

If $f: A \rightarrow B$ is a \mathcal{C}_n -morphism and $f \neq 0$, then there exists an $x \in A$ such that $f(x) \neq 0$. Define $g: \mathbb{K} \rightarrow A$ by $g(t) = tx$. Then $f \circ g \neq 0$. So \mathbb{K} is a generator for \mathcal{C}_n . QED on D1b

Now to continue the proof of D1 and D3. Note that B4, B6, and D1a insure that \mathcal{E} is colocally small and \mathcal{F} is locally small. The fact that $\mathbb{K} \in \text{ob}\mathcal{E} \cap \text{ob}\mathcal{F}$ and that \mathcal{E} and \mathcal{F} are full, together with D1b insure that \mathbb{K} is a cogenerator of \mathcal{F} and a generator of \mathcal{E} . By proposition 2 of §2.6 of [51], \mathcal{F} is complete and \mathcal{E} is cocomplete. By corollary 4 of §2.6 of [51], the inclusion functors from \mathcal{E} to \mathcal{C}_n and \mathcal{F} to \mathcal{C} preserve colimits and limits respectively. So by theorem 2 of §2.11 of [51], \mathcal{E} is coreflective in \mathcal{C}_n and \mathcal{F} is reflective in \mathcal{C} . By 13.1.1 of [18], \mathcal{E} is bicoreflective in \mathcal{C}_n and \mathcal{F} is bireflective in \mathcal{C} .

But epimorphisms are surjections in \mathcal{C}_n , epimorphisms have dense images in \mathcal{C} , and monomorphisms are injections in both cases (cf. B4, B5, and B6). Thus in D1, the canonical morphism from the coreflection to the space is

bijjective and continuous; and in D3, the canonical map from the space to the reflection is a continuous, injective map with a dense image. QED on D1 & D3

D5 Corollary to D3. If $F \in \text{ob } \mathcal{C}$, then \widehat{F} and $i: F \rightarrow \widehat{F}$ of D3 may be chosen so that

1) there exists a locally convex topology ξ on F' and a collection \mathcal{G} of balanced, convex, ξ -compact, equicontinuous subsets of F' such that $\widehat{F} = (F', \xi)'_{\mathcal{G}}$, where the subscript \mathcal{G} denotes the topology of uniform convergence on the elements of \mathcal{G} ; and

2) $i: F \rightarrow \widehat{F}$ is just the evaluation map.

Proof: Let \widehat{F} and $i: F \rightarrow \widehat{F}$ be as they are obtained from D3. Thus ${}^t i: (\widehat{F})' \rightarrow F'$ is an isomorphism of vector spaces. So if $(\widehat{F})'$ is given the $\sigma((\widehat{F})', \widehat{F})$ topology, then there exists a unique topology ξ on F' such that ${}^t i$ is a \mathcal{C} -isomorphism. Let \mathcal{G} equal the image under ${}^t i$ of all balanced, convex, $\sigma((\widehat{F})', \widehat{F})$ -compact, equicontinuous subsets of $(\widehat{F})'$.

The rest of the proof proceeds in much the same way as the proof of 2.13. QED

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\mathcal{O}	(category)	1.4
\mathcal{O}_n	(category)	1.4
a	(natural transformation)	18.3
a'	(natural transformation)	23.6
\mathcal{E}	(category)	1.12
$b(E, F)$	(functor)	17.1
$b(E, F; G)$	(functor)	17.1
$b_p(E, F)$	(functor)	17.1
$b_p(E, F; G)$	(functor)	17.1
$B(E, F)$	(functor)	17.1
$B(E, F; G)$	(functor)	17.1
\mathcal{C}	(category)	1.1
\mathcal{C}_n	(category)	1.1
c	(natural transformation)	18.3
c'	(natural transformation)	23.6
$c(X)$	(functor)	10.1
$c_C(X)$	(functor)	10.1
$C(X)$	(functor)	10.1
$c(X, E)$	(functor)	10.1
$c_C(X, E)$	(functor)	10.1
$C(X, E)$	(functor)	10.1

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$\text{hom}^{\text{op}}(E,F)$ (functor) 3.1

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